MATH1050/1058 Proof-writing Exercise 7 (Answers and selected solutions)

1. (a) **Answer.**

   There exist some $x, y, z \in \mathbb{N}$ such that $x + y$, $y + z$ are divisible by 3 and$x + z$ is not divisible by 3.

   (b) **Solution.**

   Take $x = z = 1$, $y = 2$.

   We have $x, y, z \in \mathbb{N}$.

   Note that $x + y = y + z = 3 = 1 \cdot 3$. We have $1 \in \mathbb{Z}$.

   Then, by definition, $x + y$, $y + z$ are divisible by 3.

   Note that $x + z = 2$. We verify that 2 is not divisible by 3:

   Suppose 2 were divisible by 3.

   Then there would exist some $k \in \mathbb{Z}$ such that $2 = 3k$.

   For the same k, we would have $k = \dfrac{2}{3}$. Then k is not an integer.

   Contradiction arises.

2. **Answer.**

   (a) One possible counter-example is given by: $x = y = 10$, and $z = 5$.

   (b) One possible counter-example is given by: $x = 1$, $y = 2$.

   (c) One possible counter-example is given by: $s = \sqrt{2}$, $t = -\sqrt{2}$.

   (d) One possible counter-example is given by: $a = 8$, $b = 9$, $c = 6$.

   (e) One possible counter-example is given by: $n = 3$, $\zeta = \cos\left(\dfrac{2\pi}{9}\right) + i\sin\left(\dfrac{2\pi}{9}\right)$.

   (f) One possible counter-example is given by: $n = 3$, $\zeta = \cos\left(\dfrac{2\pi}{9}\right) + i\sin\left(\dfrac{2\pi}{9}\right)$.

3. (a) **Answer.**

   There exist some sets A, B, C such that $A \backslash (C \backslash B) \not\subset A \cap B$.

   (b) **Solution.**

   Regard $0, 1, 2$ as distinct objects.

   Let $A = \{0, 1\}$, $B = \{1\}$, $C = \{2\}$.

   We have $A \cap B = B = \{1\}$, $C \backslash B = C = \{2\}$, $A \backslash (C \backslash B) = A = \{0, 1\}$.

   Note that $0 \in A \backslash (C \backslash B)$ and $0 \notin A \cap B$.

   Hence $A \backslash (C \backslash B) \not\subset A \cap B$.

4. **Answer.**

   (a) One possible counter-example is given by: $A = \{0\}$, $B = \{1\}$, $C = \{2\}$.

   (b) One possible counter-example is given by: $A = \{0\}, B = \{1\}, C = \{2\}$.

   (c) One possible counter-example is given by: $A = \{0\}$ and $B = C = \{0, 1\}$.

   (d) One possible counter-example is given by: $A = \{1\}, B = \{2\}, C = \{0, 1, 2\}$.

   (e) One possible counter-example is given by: $A = \{0, 2\}$, $B = \{1\}$, $C = \{1, 2\}$.

   (f) One possible counter-example is given by: $A = \{1\}$, $B = \{2\}$, $C = \{0, 1\}$, $D = \{0, 2\}$.

5. **Answer.**

   (a) One possible counter-example is given by: $I = \mathbb{R}$, $f : I \longrightarrow \mathbb{R}$ given by $f(x) = x^3$ for any $x \in I$.

   (b) One possible counter-example is given by: $I = \mathbb{R}$, $f : I \longrightarrow \mathbb{R}$ given by $f(x) = 0$ for any $x \in \mathbb{R}$.

   (c) One possible counter-example is given by: $I = (0, 1)$, $J = (1, 2)$,

   $$f : I \cup J \longrightarrow \mathbb{R} \text{ given by } f(x) = \begin{cases} 0 & \text{if} \quad x \in I \\ 1 & \text{if} \quad x \in J \end{cases}$$

6. **Answer.**

   (a) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (b) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = A$.

   (c) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

   (d) One possible counter-example is given by: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

7. (a) **Solution.**

   [We want to prove this statement: '*Suppose S is a subset of* $\mathbb{R}$. *Further suppose λ, μ are greatest elements of S. Then $\lambda = \mu$.*']

   Suppose S is a subset of $\mathbb{R}$. Further suppose λ, μ are greatest element of S.

   By definition of greatest element, we have $x \leq \lambda$ for any $x \in S$. Also by definition of greatest element, $\mu \in S$. Then $\mu \leq \lambda$.

   Modifying the argument above (by interchanging the roles of λ, μ), we have $\lambda \leq \mu$.

   We have $\mu \leq \lambda$ and $\lambda \leq \mu$. Then $\lambda = \mu$.

   (b) *Comment.*

   The statement to be proved should be formulated as:

   - *Let ζ be a complex number. Suppose ζ is neither real nor purely imaginary. Let z be a complex number. Let a, b, c, d be real numbers. Suppose $z = a\zeta + b\overline{\zeta}$ and $z = c\zeta + d\overline{\zeta}$. Then $a = c$ and $b = d$.*

   The argument should start in this way:

   Let ζ be a complex number. Suppose ζ is neither real nor purely imaginary.

   Pick any complex number z. Let a, b, c, d be real numbers. Suppose $z = a\zeta + b\overline{\zeta}$ and $z = c\zeta + d\overline{\zeta}$.

   (c) *Comment.*

   The statement to be proved should be formulated as:

   - *Let p be a positive real number, and q be a real number. Suppose $f(x)$ be the cubic polynomial given by $f(x) = x^3 + px + q$.*
     Let v be a real number. Let α, β be real numbers. Suppose '$u = \alpha$', '$u = \beta$' are real solutions of the equation $f(u) = v$ with unknown u.
     Then $\alpha = \beta$.

8. **Solution.**

   [We want to prove this statement: '*Let I be an interval in* $\mathbb{R}$, *and $f, g : I \longrightarrow \mathbb{R}$ be functions. Suppose f is strictly increasing on I and g is strictly decreasing on I.*
   Let $c, c' \in I$. Suppose $f(c) = g(c)$ and $f(c') = g(c')$. Then $c = c'$.']

   Let I be an interval in $\mathbb{R}$, and $f, g : I \longrightarrow \mathbb{R}$ be functions. Suppose f is strictly increasing on I and g is strictly decreasing on I.

   Pick any $c, c' \in I$. Suppose $f(c) = g(c)$ and $f(c') = g(c')$. We verify that $c = c'$ by the proof-by-contradiction method:

   - Suppose it were true that $c \neq c'$.
     Without loss of generality, assume $c < c'$.
     Since f is strictly increasing on I, we would have $f(c) < f(c')$.
     Since g is strictly decreasing on I we would have $g(c) > g(c')$.
     Recall that $f(c) = g(c)$ and $f(c') = g(c')$.
     Then $f(c) < f(c') = g(c') < g(c) = f(c)$. Therefore $f(c) < f(c)$. Contradiction arises.

9. **Solution.**

[We want to prove this statement: 'For any $\mathbf{v} \in \mathbb{R}^n$, for any $c_1, c_2, \cdots, c_k, d_1, d_2, \cdots, d_k \in \mathbb{R}$, if $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ and $\mathbf{v} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k$ then $c_1 = d_1$, $c_2 = d_2$, ... and $c_k = d_k$.']

Pick any $\mathbf{v} \in \mathbb{R}^n$.

Let $c_1, c_2, \cdots, c_k, d_1, d_2, \cdots, d_k \in \mathbb{R}$.

Suppose that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$ and $\mathbf{v} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k$.

Then $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{v} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k$.

Therefore $(c_1 - d_1)\mathbf{u}_1 + (c_2 - d_2)\mathbf{u}_2 + \cdots + (c_k - d_k)\mathbf{u}_k = \mathbf{0}$.

Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent, we have $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$.

Then $c_1 = d_1$, $c_2 = d_2$, ..., and $c_k = d_k$.

10. ——

11. ——