MATH1050/1058 Proof-writing Exercise 6

Advice.

- All the questions are concerned with arguments in set language.
- Study the Handout *Set operations*, *Examples of proofs for properties of basic set operations*, *Power set* before answering the questions.
- Besides the handouts mentioned above, Question (7) of Assignment 6 may also be relevant.
- When giving an argument, remember to adhere to definition, always.

1. (a) Explain the word *subset* by stating an appropriate definition.

   (b) Explain the phrase *union of two sets* by stating its appropriate definition.

   (c) Prove the statement ($\sharp$) below, with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

   ($\sharp$) *Let A, B, C, D be sets. Suppose $A \subset C$ and $B \subset D$. Then $A \cup B \subset C \cup D$.*

2. (a) Formulate the definition for the notion of *set equality* in terms of *subset relation.*

   (b) Prove the statement ($\sharp$), with reference to the definitions for the notions of *set equality* and *subset relation.*

   ($\sharp$) *Let A, B, C be sets. Suppose $A \subset B$, $B \subset C$, and $C \subset A$. Then $A = B$.*

3. (a) Explain the phrases *intersection of two sets* and *complement of a set in another (not distinct) set* by stating their appropriate definitions.

   (b) Prove the statements below, with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

   i. *Let A, B be sets. Suppose $A \subset A \backslash B$. Then $A \cap B = \emptyset$.*

   ii. *Let A, B be sets. Suppose $A \cap B = \emptyset$. Then $A \subset A \backslash B$.*

   (c) Prove the statements ($\natural$), with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

   ($\natural$) *Suppose A, B are sets. Then $A \subset A \cap B$ iff $A \backslash B = \emptyset$.*

4. Prove the statements below, with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

   (a) *Suppose A, B are sets. Then $(A \cup B) \backslash A = B \backslash (A \cap B)$.*

   (b) *Suppose A, B, C are sets. Then $A \backslash (B \backslash C) = (A \backslash B) \cup (A \cap C)$.*

   (c) *Let A, B be sets. Suppose $A \cup B = A \cap B$. Then $A = B$.*

   (d)$^\diamond$ *Let A, B, C be sets. Suppose $A \subset C$ and $B \subset C$. Then $(C \backslash A) \backslash (C \backslash B) = B \backslash A$.*

5. Prove the statements below:

   (a) *Suppose A, B are sets. Then $A \subsetneq B$ iff ($A \subset B$ and $B \not\subset A$).*

   (b) *Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Further suppose $A \subsetneq B$ or $B \subsetneq C$. Then $A \subsetneq C$.*

6. Prove the statements below:

   (a) *Suppose A, B are sets. Then A, B are disjoint iff the statement ($\sharp$) holds:*

   ($\sharp$) *For any object x, if $x \in A$ then $x \notin B$.*

   Remark. Hence we have a re-formulation for the notion of *disjointness* for sets.

   (b)$^\diamond$ *Let A, B, C, D be sets. Suppose A, B are disjoint. Further suppose C is a subset of A, and D is a subset of B. Then C, D are disjoint.*

   (c) *Let A, B, S be sets. Suppose A, B are disjoint. Then $S \cap A, S \cap B$ are disjoint.*

7. (a) Explain the phrase *power set of a set* by stating its appropriate definition.

   (b)$^\diamondsuit$ Prove the statement ($\sharp$):

   ($\sharp$) *Let A, B be sets. Suppose $\mathfrak{P}(B) \in \mathfrak{P}(A)$. Then $S \in A$ for any subset S of B.*

8. Let M be a set, and C be a subset of $\mathfrak{P}(M)$.

   Define $I = \{x \in M : x \in V \text{ for any } V \in C\}$, $J = \{x \in M : x \in V \text{ for some } V \in C\}$.

   Prove the statements below:

   (a)$^\diamondsuit$ *Let P be a subset of M. Suppose $P \subset V$ for any $V \in C$. Then $P \subset I$.*

   (b)$^\diamondsuit$ *Let Q be a subset of M. Suppose $V \subset Q$ for any $V \in C$. Then $J \subset Q$.*

   (c)$^\clubsuit$ *Let R be a subset of M. Suppose $D = \{V \cap R \mid V \in C\}$, and $K = \{x \in M : x \in U$ for some $U \in D\}$. Then $K = J \cap R$.*