MATH1050/1058 Proof-writing Exercise 3

Advice.

- Most of the questions are concerned with the method of proof-by-contradiction. Some are concerned with the handling of of '*there exists*'.
- When doing proofs, remember to adhere to definition, always.

  Study the handouts *Basic results on divisibility*, and *Rationals and irrationals*.
- Besides the handout mentioned above, Question (3), Question (4), in Assignment 2 and Question (10) in Assignment 3 are also suggestive on what it takes to give the types of argument meant to be written here, and on the level of rigour required.

1. Apply proof-by-contradiction to justify the statements below:

   (a) *Let a, b be complex numbers. Suppose $a^4 + a^3b + a^2b^2 + ab^3 + b^4 \neq 0$. Then at least one of a, b is non-zero.*

   (b) *Let a be a real number and b be a complex number. Suppose $a^3|b| > 2$. Then $a^6 + 9|b|^2 > 6$.*

   (c) *Let ζ be a complex number. Suppose that $|\zeta| \leq \varepsilon$ for any positive real number ε. Then $\zeta = 0$.*

2.$^\diamond$ *In this question, take for granted that $\sqrt{2}, \sqrt{3}$* are irrational numbers*.*

   Apply proof-by-contradiction to justify the statements below:

   (a) *$\sqrt{2} + \sqrt{3}$ is an irrational number.*

   Remark. *Hint.* Write $r = \sqrt{2} + \sqrt{3}$. Can you re-express one of $\sqrt{2}, \sqrt{3}$ as a fractional expression whose numerator and denominator involve only integers and the non-negative integral powers of r?

   (b) *$\sqrt{3} - \sqrt{2}$ is an irrational number.*

   Remark. See if you can generalize the argument to prove the statement (♯):

   (♯) *Suppose a, b are non-zero rational numbers. Then $a\sqrt{2} + b\sqrt{3}$ is an irrational number.*

3. Take for granted the validity of Euclid's Lemma where appropriate and necessary. You may also take for granted that $2, 3, 5$ are prime numbers.

   Apply proof-by-contradiction to justify the statements below:

   (a) *$\sqrt{3}$ is an irrational number.*

   (b) *$\sqrt[3]{5}$ is an irrational number.*

   (c) *$\sqrt[3]{4}$ is an irrational number.*

4. Apply proof-by-contradiction to justify the statements below:

   (a) *2 is not divisible by 3.*

   Remark. Apply the definition for the notion of divisibility to obtain an equality with 2 on one side and an expression involving 3 and some intger on the other side. Then obtain a contradiction by considering the magnitudes of the integers involved.

   (b)$^\diamond$ *3 is not divisible by 2.*

   (c)$^\clubsuit$ *$\sqrt{6}$ is irrational.*

   Remark. Take for granted the validity of Euclid's Lemma where appropriate and necessary. You may also need the results described in the previous parts.

5. We recall/introduce the definitions for the notions of *algebraicity* and *transcendence* for complex numbers:

   - *Let α be a complex number. We say that α is* **algebraic** *if there exists some non-constant polynomial $f(x)$ whose coefficients are rational numbers such that $f(\alpha) = 0$.*
   - *Let τ be a complex number. We say that τ is* **transcendental** *if τ is not algebraic.*

(a) Prove the statements below:

i.$^\diamondsuit$ *Let α be a positive real number. Suppose α is algebraic. Then $\sqrt[3]{\alpha}$ is algebraic.*

ii.$^\clubsuit$ *Let α be a non-zero complex number. Suppose α is algebraic. Then $\dfrac{1}{\alpha}$ is algebraic.*

iii.$^\heartsuit$ *Let α be a complex number. Suppose α is algebraic. Then α^2 is algebraic.*

(b) Prove the statements below:

i. *Let τ be a positive real number. Suppose τ is transcendental. Then τ^3 is transcendental.*

ii. *Let τ be a non-zero complex number. Suppose τ is transcendental. Then $\dfrac{1}{\tau}$ is transcendental.*

iii. *Let τ be a positive real number. Suppose τ is transcendental. Then $\sqrt{\tau}$ is transcendental.*

6.$^\heartsuit$ Let α, β be complex numbers, a_0, a_1, b_0, b_1 are rational numbers, and $f(x) = x^2 + a_1x + a_0$, $g(x) = x^2 + b_1x + b_0$. Suppose $f(\alpha) = 0$ and $g(\beta) = 0$.

Define $\gamma = \alpha\beta$.

(a) Express $\gamma\alpha$ in the form $A_1+A_2\gamma+A_3\alpha+A_4\beta$. Here A_1, A_2, A_3, A_4 are appropriate rational numbers, possibly given in terms of a_0, a_1, b_0, b_1.

(b) Express $\gamma\beta$ in the form $B_1+B_2\gamma+B_3\alpha+B_4\beta$. Here B_1, B_2, B_3, B_4 are appropriate rational numbers, possibly given in terms of a_0, a_1, b_0, b_1.

(c) Express γ^2 in the form $C_1+C_2\gamma+C_3\alpha+C_4\beta$. Here C_1, C_2, C_3, C_4 are appropriate rational numbers, possibly given in terms of a_0, a_1, b_0, b_1.

(d) Express γ^3 in the form $D_1 + D_2\gamma + D_3\gamma^2 + D_4\alpha + D_5\beta$. Here D_1, D_2, D_3, D_4, D_5 are appropriate rational numbers, possibly given in terms of a_0, a_1, b_0, b_1.

(e) Express γ^4 in the form $E_1 + E_2\gamma + E_3\gamma^2 + E_4\gamma^3 + E_5\alpha + E_6\beta$. Here $E_1, E_2, E_3, E_4, E_5, E_6$ are appropriate rational numbers, possibly given in terms of a_0, a_1, b_0, b_1.

(f) Prove that there exist some rational numbers c_0, c_1, c_2, c_3 such that $\gamma^4 + c_3\gamma^3 + c_2\gamma^2 + c_1\gamma + c_0 = 0$.

(g) Prove that γ is algebraic.

(*Hint.* The work in the previous parts constitute the relevant roughwork for the argument for this part.)

Remark. Using a similar argument, we can also prove that $\alpha + \beta$ is algebraic.

What is described above is a 'baby case' for a more general result:

($\sharp$) *For any $\alpha, \beta \in \mathbb{C}$, if α, β are algebraic then $\alpha + \beta$, $\alpha\beta$ are algebraic.*

With the result ($\sharp$), we can deduce that the set of all algebraic numbers constitute a field.

7. For each $n \in \mathbb{N}\backslash\{0\}$, define $A_n = \displaystyle\sum_{j=1}^{n} \frac{1}{j}$, $B_n = \displaystyle\sum_{k=1}^{n} \frac{1}{2k}$, $C_n = \displaystyle\sum_{k=1}^{n} \frac{1}{2k-1}$.

(a) i. Prove that $B_n = \dfrac{1}{2}A_n$ and $C_n = A_{2n} - \dfrac{1}{2}A_n$ for any $n \in \mathbb{N}\backslash\{0\}$.

ii. Prove that $C_n - B_n \geq \dfrac{1}{2}$ for any $n \in \mathbb{N}\backslash\{0, 1\}$.

(b) By applying proof-by-contradiction, or otherwise, prove that $\{A_n\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}$.

Remark. Take for granted the result ($\star$) about inequality for limits of infinite sequences:

($\star$) *Let $\{x_n\}_{n=0}^{\infty}$ be an infinite sequence of real number, and t be a real number. Suppose $x_n \geq t$ for any $n \in \mathbb{N}$. Also suppose $\{x_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$. Then $\lim\limits_{n\to\infty} x_n \geq t$.*