MATH1050/1058 Proof-writing Exercise 2

Advice.

- All the questions are concerned with the handling of '*there exists*' and/or with proof-by-contradiction argument.
- When doing proofs, remember to adhere to definition, always.
  Study the handouts *Basic results on divisibility*, and *Rationals and irrationals*.
- Besides the handout mentioned above, Question (1), Question (2), Question (5) in Assignment 2 are also suggestive on what it takes to give the types of argument meant to be written here, and on the level of rigour required.

1. Prove the statements below (with direct reference to the definition of *rational numbers*):
   (a) *Let x, y be real numbers. Suppose x, y are rational. Then $x - y$ is rational.*
   (b) *Let x, y be real numbers. Suppose x, y are rational and $y \neq 0$. Then $\dfrac{x}{y}$ is rational.*

2. Prove the statements below (with direct reference to the definition of *divisibility*):
   (a) *Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $|x| = |y|$.*
   (b) *Let $x, y, z \in \mathbb{Z}$. Suppose x is divisible by y^2, and y is divisible by z^3. Then x is divisible by z^6.*
   (c)$^\diamondsuit$ *Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n. Then for any $y \in \mathbb{Z}$, $(3x^2 + 4y)^5 + (3x^2 - 4y)^5$ is divisible by $6n^2$.*
   Remark. Where appropriate, make good use of what you know about geometric progressions (or the Binomial Theorem).

3. Prove the statements below (with direct reference to the definition of *arithmetic progressions* and *geometric progressions*):
   (a) *Suppose $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ are arithmetic progressions. Then $\{a_n + b_n\}_{n=0}^{\infty}$ is an arithmetic progression.*
   (b)$^\diamondsuit$ *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of integers. Suppose $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression. Then, for any non-zero complex number ζ, the infinite sequence $\{\zeta^{a_n}\}_{n=0}^{\infty}$ is a geometric progression.*

4. Prove the statements below (with direct reference to *boundedness*):
   (a)$^\diamondsuit$ *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$. Then, for any positive real number c, $\{ca_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$.*
   (b)$^\diamondsuit$ *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{a_n\}_{n=0}^{\infty}$ is bounded in $\mathbb{R}$. Then $\{a_n + a_{2n}\}_{n=0}^{\infty}$ is bounded in $\mathbb{R}$.*
   (c)$^\clubsuit$ *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{a_n\}_{n=0}^{\infty}$ is bounded in $\mathbb{R}$. Then $\{a_n - a_{2n}\}_{n=0}^{\infty}$ is bounded in $\mathbb{R}$.*

5.$^\clubsuit$ We introduce (or recall from your *calculus* course) the definitions on *relative extremum* for real-valued functions of one real variable.

   Let $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D. Suppose $p \in D$.
   - *h is said to be attain a* **relative maximum** *at p if the statement* (RelMax) *holds:*
     (RelMax) *There exists some positive real number δ such that for any $x \in D$, if $|x - p| < \delta$ then $h(x) \leq h(p)$.*
   - *h is said to be attain a* **relative minimum** *at p if the statement* (RelMin) *holds:*
     (RelMin) *There exists some positive real number δ such that for any $x \in D$, if $|x - p| < \delta$ then $h(x) \geq h(p)$.*

   Prove the statement ($\sharp$):

   ($\sharp$) *Let I be an open interval, and $f : I \longrightarrow \mathbb{R}$ be a real-valued function with one real variable. Suppose $a, c, b \in I$ and $a < c < b$. Further suppose f is strictly decreasing on $[a, c]$ and f is strictly increasing on $[c, b]$.*
   Then f attains a relative minimum at c.

   Remark. The statement ($\sharp$), when combined with consequences of the Mean-Value Theorem which link up strict monotonicity and the 'signs' of the first derivative, give rise to the *First Derivative Test (on relative minimum)* that you learnt in your *calculus* course. There are also the corresponding statements and results concerned with relative maximum.

   Further remark. The converse of the statement ($\sharp$) is false.