MATH1050/1058 Guided Study Exercise 1 (Answers)

1. ——
2. ——
3. ——
4. **Solution.**

   (a) Let n be a positive integer, and $f(x)$ be the polynomial $f(x) = (1+x)^n$.

   Note that $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$ as polynomials.

   i. $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k = f(1) = (1+1)^n = 2^n$.

   ii. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = f(-1) = (1-1)^n = 0$.

   iii. $\sum_{k=0}^{n} \frac{1}{2^k} \binom{n}{k} = f(\frac{1}{2}) = \left(1 + \frac{1}{2}\right)^n = \frac{3^n}{2^n}$.

   iv. $\sum_{k=0}^{n} \frac{(-1)^k \cdot 3^{k-1}}{5^{k+1}} \binom{n}{k} = \frac{1}{15} \sum_{k=0}^{n} \frac{(-1)^k \cdot 3^k}{5^k} \binom{n}{k} = \frac{1}{15} f(-\frac{3}{5}) = \frac{1}{15}\left(1 - \frac{3}{5}\right)^n = \frac{2^n}{15 \cdot 5^n}$.

   (b) Let m be a positive integer. Then $2m$ is a positive integer.
   Let $g(x)$ be the polynomial $g(x) = (1+x)^{2m}$.

   Note that $g(x) = \sum_{k=0}^{2m} \binom{2m}{k} x^k$ as polynomials.

   i. $\sum_{k=0}^{2m} \binom{2m}{k} = g(1) = 2^{2m}$.

   ii. $\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} = g(-1) = 0$.

   iii.
   $$\begin{aligned}\sum_{k=0}^{m} \binom{2m}{2k} &= \sum_{j=0}^{2m} \frac{1}{2}\left(\binom{2m}{j} + (-1)^j \binom{2m}{j}\right) \\ &= \frac{1}{2}\left(\sum_{j=0}^{2m} \binom{2m}{j} + \sum_{j=0}^{2m} (-1)^j \binom{2m}{j}\right) = \frac{1}{2}(2^{2m} + 0) = 2^{2m-1}\end{aligned}$$

   iv.
   $$\begin{aligned}\sum_{k=0}^{m-1} \binom{2m}{2k+1} &= \sum_{j=0}^{2m} \frac{1}{2}\left(\binom{2m}{j} - (-1)^j \binom{2m}{j}\right) \\ &= \frac{1}{2}\left(\sum_{j=0}^{2m} \binom{2m}{j} - \sum_{j=0}^{2m} (-1)^j \binom{2m}{j}\right) = \frac{1}{2}(2^{2m} - 0) = 2^{2m-1}\end{aligned}$$

   Answer.

   (a) i. 2^n.
   ii. 0.
   iii. $\frac{3^n}{2^n}$.
   iv. $\frac{2^n}{15 \cdot 5^n}$.

(b) i. 2^{2m}.

ii. 0.

iii. 2^{2m-1}.

iv. 2^{2m-1}.

5. **Solution.**

(a) Let $n \in \mathbb{N}\backslash\{0\}$, and $k \in \mathbb{Z}$.

- (Case 1.) Suppose $0 < k \leq n$. Then

$$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k! \cdot (n-k)!} = \frac{n!}{(k-1)! \cdot (n-k)!} = n \cdot \frac{(n-1)!}{(k-1)! \cdot [(n-1)-(k-1)]!} = n \cdot \binom{n-1}{k-1}.$$

- (Case 2.) Suppose $k \leq 0$ or $k > n$. Then $k \cdot \binom{n}{k} = 0 = n \cdot \binom{n-1}{k-1}$.

Hence in any case, $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

(b) i. Let n be a positive integer.

$$\sum_{k=0}^{n} k \binom{n}{k} = \sum_{k=1}^{n} k \binom{n}{k} = \sum_{k=1}^{n} n \binom{n-1}{k-1} = n \sum_{k=1}^{n} \binom{n-1}{k-1} = n \sum_{j=0}^{n-1} \binom{n-1}{j} = n \cdot 2^{n-1}.$$

(c) i. Let m be a positive integer.

$$\begin{aligned}
\sum_{k=0}^{m} \frac{1}{k+1} \binom{m}{k} &= \sum_{k=0}^{m} \frac{1}{m+1} \binom{m+1}{k+1} \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k+1} \\
&= \frac{1}{m+1} \sum_{j=1}^{m+1} \binom{m+1}{j} \\
&= \frac{1}{m+1} \left(\sum_{j=0}^{m+1} \binom{m+1}{j} - 1 \right) = \frac{2^{m+1}-1}{m+1}
\end{aligned}$$

Answer.

(a) ——

(b) i. $n \cdot 2^{n-1}$

ii. When $n = 1$, $\sum_{k=0}^{n} k(k-1) \binom{n}{k} = 1$. Whenever $n \geq 2$, $\sum_{k=0}^{n} k(k-1) \binom{n}{k} = 0$.

iii. $n(n-1) \cdot 2^{n-2}$

iv. $n(n+1) \cdot 2^{n-2}$

(c) i. $\dfrac{2^{m+1}-1}{m+1}$

ii. $\dfrac{1}{m+1}$

iii. $\dfrac{2^{m+2}-m-3}{(m+2)(m+1)}$

iv. $\dfrac{2^{m+1}m+1}{(m+2)(m+1)}$

6. (a) **Answer.**

(I) There exists some non-zero complex number r

(II) for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} = r$

(b) **Answer.**

(I) there exists some non-zero complex number r such that for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} = r$

(II) $\dfrac{b_1}{b_0} = r$

(III) $\dfrac{b_m}{b_{m-1}} = r$

(IV) $\dfrac{b_1}{b_0} \cdot \dfrac{b_2}{b_1} \cdot \dfrac{b_3}{b_2} \cdot \ldots \cdot \dfrac{b_{m-1}}{b_{m-2}} \cdot \dfrac{b_m}{b_{m-1}} = r^m$

(V) $b_m = b_0 r^m$

(c) **Solution.**

Let $\{a_n\}_{n=0}^{\infty}$ be a geometric progression. Suppose $k, \ell, m \in \mathbb{N}$, and $a_k = A$, $a_\ell = B$ and $a_m = C$.

By the result in the previous part, there exists some non-zero complex number r such that for any $n \in \mathbb{N}$, $a_n = a_0 r^n$.

In particular, $a_k = a_0 r^k$, $a_\ell = a_0 r^\ell$ and $a_m = a_0 r^m$.

Then

$$\begin{aligned} A^{\ell-m} B^{m-k} C^{k-\ell} &= (a_0 r^k)^{\ell-m} (a_0 r^\ell)^{m-k} (a_0 r^m)^{k-\ell} \\ &= {a_0}^{(\ell-m)+(m-k)+(k-\ell)} r^{k(\ell-m)+\ell(m-k)+m(k-\ell)} \\ &= {a_0}^0 r^0 = 1 \end{aligned}$$

7. (a) **Answer.**

(I) Suppose a, b, c are in arithmetic progression.

(II) d

(III) $c - b = d$

(IV) $b^2 - a^2 - ca + bc = (b-a)(b+a) + (b-a)c = d(a+b+c)$

(V) $[(c^2 - ab) - (b^2 - ca)] = c^2 - b^2 - ab + ca = (c-b)(c+b) + (c-b)a = d(a+b+c)$

(VI) $(b^2 - ca) - (a^2 - bc) = (c^2 - ab) - (b^2 - ca)$

(b) **Solution.**

Let a, b, c be complex numbers. Suppose $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression. Further suppose $a + b + c \neq 0$.

By assumption, $b^2 - ca = \dfrac{(a^2 - bc) + (c^2 - ab)}{2}$.

Therefore $2b^2 - 2ca = a^2 + c^2 - ab - bc$.

Hence $0 = (a^2 - b^2) + (c^2 - b^2) + (ac - ab) + (ac - bc) = \cdots = (a + c - 2b)(a + b + c)$.

Since $a + b + c \neq 0$, we have $a + c - 2b = 0$. Then $b = \dfrac{a+b}{2}$.

Therefore a, b, c are in arithmetic progression.

8. ——

9. ——

10. **Solution.**

Let n be a positive integer.

(a) Pick any $x, t \in \mathbb{R}$. Suppose $t \neq 0$.

We have

$$\begin{aligned} \frac{(x+t)^n - x^n}{t} &= \frac{[(x+t) - x][(x+t)^{n-1} + (x+t)^{n-2}x + (x+t)^{n-3}x^2 + \cdots + (x+t)x^{n-2} + x^{n-1}]}{t} \\ &= (x+t)^{n-1} + (x+t)^{n-2}x + (x+t)^{n-3}x^2 + \cdots + (x+t)x^{n-2} + x^{n-1} \end{aligned}$$

(b) Pick any $x, t \in \mathbb{R}$. Suppose $t \neq 0$.
We have

$$\frac{(x+t)^n - x^n}{t} = (x+t)^{n-1} + (x+t)^{n-2}x + (x+t)^{n-3}x^2 + \cdots + (x+t)x^{n-2} + x^{n-1}$$

As $t \longrightarrow 0$, we have $(x+t)^j \longrightarrow x^j$ for each $j = 1, 2, \cdots, n-1$.
Then $\dfrac{(x+t)^n - x^n}{t} \longrightarrow \underbrace{x^{n-1} + x^{n-2} \cdot x + x^{n-3} \cdot x^2 + \cdots + x \cdot x^{n-2} + x^{n-1}}_{n \text{ copies}} = nx^{n-1}$.

(c) Pick any $x, t \in \mathbb{R}$. Suppose $x > 0$, $t \neq 0$ and $x + t > 0$.
We have

$$\begin{aligned} \frac{1}{t}\left[\frac{1}{(x+t)^n} - \frac{1}{x^n}\right] &= -\frac{1}{x^n(x+t)^n} \cdot \frac{(x+t)^n - x^n}{t} \\ &= -\frac{1}{x^n(x+t)^n} \cdot [(x+t)^{n-1} + (x+t)^{n-2}x + (x+t)^{n-3}x^2 + \cdots + (x+t)x^{n-2} + x^{n-1}] \end{aligned}$$

As $t \longrightarrow 0$, we have $(x+t)^j \longrightarrow x^j$ for each $j = 1, 2, \cdots, n-1, n$.
Then $\dfrac{1}{t}\left[\dfrac{1}{(x+t)^n} - \dfrac{1}{x^n}\right] \longrightarrow -\dfrac{1}{x^n \cdot x^n} \cdot \underbrace{(x^{n-1} + x^{n-2} \cdot x + x^{n-3} \cdot x^2 + \cdots + x \cdot x^{n-2} + x^{n-1})}_{n \text{ copies}} = -\dfrac{n}{x^{n+1}}$.

11. **Answer.**

(a) $A = B = C = D = E = F = G = H = I = J = 1$.

(b) $\dfrac{1}{(1 - \sin(\theta))(1 - x\sin(\theta))}$.