1. (a) Fill in the blanks in the passage below so as to give the definition for the notion of *identity function on a set*:

   ____(I)____ . *The* **identity function on** C *is the function* id_C : ____(II)____ defined by ____(III)____ .

   (b) Consider the statement (T):

   (T) *Let A be a set, and $f : A \longrightarrow A$ be a function. Suppose $f \circ f = f$. Further suppose (f is injective or f is surjective). Then $f = \mathsf{id}_A$.*

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (T). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > Let A be a set, and $f : A \longrightarrow A$ be a function.
   > Suppose $f \circ f = f$. —— $(\star)$
   > ____(I)____
   >
   > [We want to verify that $f = \mathsf{id}_A$. This amounts to verifying '*for any $x \in A$, $f(x) = \mathsf{id}_A(x)$*'.]
   >
   > - (Case 1.) Suppose f is injective.
   >
   >   Pick any ____(II)____ . By the definition of the function f, we have $f(x) \in A$.
   >
   >   By $(\star)$, we have $(f \circ f)(x) =$ ____(III)____.
   >
   >   By the definition of composition, we have ____(IV)____ $= f(f(x))$.
   >
   >   Then $f(f(x)) = f(x)$. Now, by ____(V)____ , we have ____(VI)____ .
   >
   >   It follows that $f = \mathsf{id}_A$.
   >
   > - (Case 2.) ____(VII)____
   >
   >   ____(VIII)____ $x \in A$. By the definition of surjectivity, ____(IX)____ .
   >
   >   Then we have $f(x) = f(f(u)) =$ ____(X)____ by the definition of composition.
   >
   >   By $(\star)$, we have ____(XI)____ $= x$. Then $f(x) = x = \mathsf{id}_A(x)$.
   >
   >   It follows that ____(XII)____ .
   >
   > Hence, in any case, $f = \mathsf{id}_A$.

   (c) Hence, or otherwise, prove the statement $(\sharp)$:

   $(\sharp)$ *Let B be a set, K be a subset of B, and $\varphi : \mathfrak{P}(B) \longrightarrow \mathfrak{P}(B)$ be the function defined by $\varphi(S) = S \cap K$ for any $S \in \mathfrak{P}(B)$. Suppose φ is injective or φ is surjective. Then $K = B$.*

2.♣ We introduce the notation for the *set of all functions from a given set to a given set*:

   Let D, R be sets. The **set of all functions with domain** D **and range** R *is denoted by* $\mathsf{Map}(D, R)$.

   Let A, B be non-empty sets. For any $x \in A$, define the function $E_x : \mathsf{Map}(A, B) \longrightarrow B$ by $E_x(f) = f(x)$ for any $x \in A$.

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (P) and a proof for the statement (Q). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) Here we prove the statement (P):

   (P) *For any $x \in A$, the function E_x is surjective.*

   > ____(I)____ . We verify that E_x is surjective:
   >
   > - ____(II)____ . Define the function $f : A \longrightarrow B$ by ____(III)____ .
   >
   >   By definition, $f \in$ ____(IV)____ . By definition of E_f, we have ____(V)____ .
   >
   >   It follows that E_x is surjective.

   (b) Here we prove the statement (Q):

(*Q*) *Suppose B has more than one element. Also suppose there exists some $u \in A$ such that E_u is injective. Then A is a singleton.*

> Suppose B has more than one element. Pick ____(I)____ $y, z \in B$.
>
> Also suppose ________(II)________ .
>
> Note that $\{u\} \subset A$. We now verify $A \subset \{u\}$:
>
> - Pick any $x \in A$. Suppose it were true that $x \notin \{u\}$. Then by definition of complement, ____(III)____ .
>
>   ____(IV)____ $f(t) = y$ ____(V)____ $t \in A$.
>
>   Define ____(VI)____ by $g(t) = \begin{cases} y & \text{if } \underline{\quad (VII) \quad} \\ z & \text{if } t \in A \backslash \{u\} \end{cases}$.
>
>   By definition, $f, g \in \mathsf{Map}(A, B)$.
>
>   We have $E_u(f) =$ ________(VIII)________ . Then, since E_u is injective, we have ____(IX)____ .
>
>   Recall that $x \in A \backslash \{u\}$. Since $f(x) = y$ and $g(x) = z$ and ____(X)____ , we have ____(XI)____ .
>
>   Now $f = g$ ____(XII)____ $f \neq g$. Contradiction arises.
>
>   It follows that in the first place, we have $x \in \{u\}$.
>
> Hence $A = \{u\}$.

3. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^4 - 4x^2$ for any $x \in \mathbb{R}$.
   i. Is f injective? Justify your answer.
   ii. Is f surjective? Justify your answer.

   (b) Verify that for any $x \in (\sqrt{2}, +\infty)$, $x^4 - 4x^2 > -4$.

   (c) Let $g : (\sqrt{2}, +\infty) \longrightarrow (-4, +\infty)$ be the function defined by $g(x) = x^4 - 4x^2$ for any $x \in (\sqrt{2}, +\infty)$.
   i. Is g injective? Justify your answer.
   ii. Is g surjective? Justify your answer.
   iii. Is g bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

4. *You are not required to prove your answers in this question.*

   The function $f : (0, +\infty) \longrightarrow J$, given by $f(x) = \dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{e^{\sqrt{x}} - e^{-\sqrt{x}}}$ for any $x \in (0, +\infty)$ is known to be a bijective function from $(0, +\infty)$ to the set J.

   (a) Express the set J explicitly as an interval.

   (b) Write down the explicit 'formula of definition' for the inverse function f^{-1} of the function f.

5. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \bar{z}$ for any $z \in \mathbb{C}$.

   (a) Verify that f is bijective.

   (b) Write down the 'formula of definition' of the inverse function of f.

6. Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.

   (a) Prove that for any $z \in \mathbb{C}$, $\dfrac{az + b}{cz + d} \neq \dfrac{a}{c}$.

   (b) Define the function $f : \mathbb{C} \backslash \{-d/c\} \longrightarrow \mathbb{C} \backslash \{a/c\}$ by $f(z) = \dfrac{az + b}{cz + d}$ for any $z \in \mathbb{C} \backslash \{-d/c\}$.
   i. Verify that f is injective.
   ii. Verify that f is surjective.
   iii. Write down the 'formula of definition' of the inverse function of f.

7. (a)$^\diamondsuit$ Let $n \in \mathbb{N} \backslash \{0\}$, and $a \in \mathbb{C} \backslash \{0\}$. Define the function $\mu : \mathbb{C} \longrightarrow \mathbb{C}$ by $\mu(z) = az^n$ for any $z \in \mathbb{C}$. Prove that μ is bijective iff $n = 1$.

   (b) Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $h(z) = \begin{cases} iz & \text{if } |z| \in \mathbb{Q} \\ \dfrac{3i}{2\bar{z}} & \text{if } |z| \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$.

i. Prove the statement (♯):

(♯) *For any $\zeta \in \mathbb{C}$, if $|\zeta|$ is irrational then $|h(\zeta)|$ is irrational.*

ii. Prove that $(h \circ h)(z) = -z$ for any $z \in \mathbb{C}$.

iii.$^\diamond$ Is h bijective? Justify your answer. (*Hint.* Make good use of the result in the previous part.)

8. (a) Fill in the blanks in the passage below so as to give the respective definitions for the notions of *relation* and *function*:

- *Let H, K, L be sets. We say that (H, K, L) is a* **relation** *if* ____(I)____ .
- *Let D, R, G be sets. We say that (D, R, G) is a* **function** *if (D, R, G) is* ____(II)____ *and the statements (E), (U) below hold:*

  (E) ____(III)____ *, there exists some* ____(IV)____ *such that* ____(V)____ .

  (U) ____(VI)____ *, if* ____(VII)____ *then* ____(VIII)____ .

  For such a function, we say that ____(IX)____ *is its domain,* ____(X)____ *is its range, and* ____(XI)____ *is its graph.*

(b) *You are not required to justify your answers in this question. In each part, you are only required to give one correct answer, although there are different correct answers.*

i. Let $A = (-1, 1]$, $B = [-2, 2)$, $G = \{(x, x) \mid x \leq 0\}$, $H = \{(x, x+1) \mid x \geq 0\}$ and $F = (A \times B) \cap (G \cup H)$. Name some appropriate $(p, q), (s, t) \in A \times B$, if such exist, for which the ordered triple $(A, B, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is a function from A to B.

ii. Let $A = [0, 2]$, $G = \{(x, x^2) \mid 0 \leq x \leq 1\}$, $H = \{(x, 3 - x) \mid 1 \leq x < 2\}$ and $F = A^2 \cap (G \cup H)$. Name some appropriate $(p, q), (s, t) \in A^2$, if such exist, for which the ordered triple $(A, A, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is an injective function from A to A.

iii. Let $A = [0, +\infty)$ and E, F be the subsets of $\mathbb{R}^2$ defined respectively by $E = \{(x, x^{-1}) \mid 0 < x \leq 1\}$, $F = \{(x, 2x^{-2}) \mid x \geq 1\}$. Name some appropriate $(m, n), (p, q) \in A^2$, if such exist, for which the ordered triple $(A, A, (E \cup F \cup \{(m, n)\}) \backslash \{(p, q)\})$ is a surjective function from A to A.

9. Let $A = [0, 4]$, $B = [4, 6]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } (x-2)^4 + 4(y-4)^2 = 16\}$. Define $f = (A, B, F)$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Here we prove the statement (A):

(A) *f is a function from A to B with graph F.*

By definition, ____(I)____ . Then f is a relation from A to B with graph F.

We verify the statement '*for any $x \in A$,* ____(II)____ *such that $(x, y) \in F$*':

- ____(III)____

  By definition, $0 \leq x \leq 4$. Then $-2 \leq x - 2 \leq 2$. Therefore $0 \leq (x-2)^4 \leq 16$.

  Hence ____(IV)____ $\leq \dfrac{16 - (x-2)^4}{4} \leq$ ____(V)____ .

  ____(VI)____ . By definition, $4 \leq y \leq 6$. Then ____(VII)____ .

  Also by definition, $(x-2)^4 + 4(y-4)^2 =$ ____(VIII)____ .

  Hence ____(IX)____ .

We verify the statement '*for any $x \in A$, for any $y, z \in B$,* ____(X)____ ':

- ____(XI)____

  Since $(x, y) \in F$, we have ____(XII)____ .

  Also, ____(XIII)____ .

  Then $(y-4)^2 =$ ____(XIV)____ $= (z-4)^2$.

  Since $y, z \in B$, we have $y - 4 \geq 0$ and $z - 4 \geq 0$.

  Then $y - 4 =$ ____(XV)____ .

  Therefore ____(XVI)____ .

It follows that f is a function.

10. Let $E = \{w \in \mathbb{C} : |w - i| < 1\}$, and $F = \{(z, w) \mid z \in \mathbb{C} \text{ and } w \in D \text{ and } (1+|z-1|)(w-i)+1 = z\}$. Define $f = (\mathbb{C}, D, F)$.

(a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Here we prove the statement (B):

(B) *f is a function from $\mathbb{C}$ to E with graph F.*

- We verify that ______(I)______ :

  ______(II)______

  Note that $|z - 1| \geq 0$. Then $1 + |z - 1| > |z - 1| \geq 0$. Therefore $1 + |z - 1| \neq 0$.

  Hence $\dfrac{z-1}{1+|z-1|}$ is ______(III)______ as a complex number.

  Moreover, $0 \leq \dfrac{|z-1|}{1+|z-1|} < 1$.—— $(*)$

  ______(IV)______ . By definition, $w \in \mathbb{C}$.

  We have $|w - i| =$ ______(V)______ < 1. (The last inequality holds by $(*)$.)

  Then ______(VI)______ .

  We have ______(VII)______ .

  Therefore $(z, w) \in F$.

- We verify that ______(VIII)______ :

  ______(IX)______ $z \in$ ______(X)______ , $w, v \in$ ______(XI)______ . ______(XII)______ .

  Then $(1 + |z - 1|)(w - i) + 1 = z$ and $(1 + |z - 1|)(v - i) + 1 = z$.

  Then $(1 + |z - 1|)(w - v) =$ ______(XIII)______ .

  We have $|z - 1| \geq 0$. Then $1 + |z - 1| > 0$. Therefore $1 + |z - 1| \neq 0$.

  Hence ______(XIV)______ . Then $w = v$.

Hence f is a function.

(b) Write down the explicit 'formula of definition' for the function f.

(c)$^{\diamondsuit}$ Verify that f is injective.

(d)$^{\clubsuit}$ Verify that f is surjective.

(e) Write down the explicit 'formula of definition' for the inverse function f^{-1} of the function f.