

1. (a) **Answer.**

(I) Suppose  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  are non-negative real numbers.

$$\text{Then } \left( \sum_{j=1}^n a_j^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \geq \left( \sum_{j=1}^n a_j b_j \right)^2.$$

(II)  $(sv - tu)^2$

(III)  $(s^2 + t^2)(u^2 + v^2)$

(IV)  $(su + tv)^2$

(V)  $a_1, a_2, \dots, a_m, a_{m+1}, b_1, b_2, \dots, b_m, b_{m+1}$  are non-negative real numbers

$$\text{(VI) } \sqrt{\sum_{j=1}^m a_j b_j}$$

(VII)  $(A^2 + a_{m+1}^2)(B^2 + b_{m+1}^2)$

(VIII)  $a_{m+1} b_{m+1}$

(IX)  $C^2$

(b) **Solution.**

Let  $n \in \mathbb{N} \setminus \{0, 1\}$ . Suppose  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  are real numbers.

Note that  $|a_1|, |a_2|, \dots, |a_n|, |b_1|, |b_2|, \dots, |b_n|$  are non-negative real numbers.

By the result in part (a), we have  $\left( \sum_{j=1}^n |a_j|^2 \right) \left( \sum_{j=1}^n |b_j|^2 \right) \geq \left( \sum_{j=1}^n |a_j b_j| \right)^2$ .

Note that  $\sum_{j=1}^n |a_j|^2 = \sum_{j=1}^n a_j^2$ ,  $\sum_{j=1}^n |b_j|^2 = \sum_{j=1}^n b_j^2$ .

By the Triangle Inequality for real numbers, we have  $\sum_{j=1}^n |a_j b_j| \geq \left| \sum_{j=1}^n a_j b_j \right|$ .

Then  $\left( \sum_{j=1}^n |a_j b_j| \right)^2 \geq \left| \sum_{j=1}^n a_j b_j \right|^2 = \left( \sum_{j=1}^n a_j b_j \right)^2$ .

Therefore  $\left( \sum_{j=1}^n a_j^2 \right) \left( \sum_{j=1}^n b_j^2 \right) = \left( \sum_{j=1}^n |a_j|^2 \right) \left( \sum_{j=1}^n |b_j|^2 \right) \geq \left( \sum_{j=1}^n |a_j b_j| \right)^2 \geq \left( \sum_{j=1}^n a_j b_j \right)^2$ .

2. (a) **Solution.**

Suppose  $x, y, z$  are real numbers.

By the Cauchy-Schwarz Inequality,  $|yz + zx + xy| \leq (y^2 + z^2 + x^2)^{\frac{1}{2}}(z^2 + x^2 + y^2)^{\frac{1}{2}} = x^2 + y^2 + z^2$ .

Then

$$\begin{aligned} & (y + z - x)^2 + (z + x - y)^2 + (x + y - z)^2 - (yz + zx + xy) \\ &= (y^2 + z^2 + x^2 + 2yz - 2xy - 2zx) + (z^2 + x^2 + y^2 + 2zx - 2xy - 2yz) \\ & \quad + (x^2 + y^2 + z^2 + 2xy - 2yz - 2zx) - (yz + zx + xy) \\ &= 3[(x^2 + y^2 + z^2) - (yz + zx + xy)] \\ &\geq 3[(x^2 + y^2 + z^2) - |yz + zx + xy|] \\ &\geq 0 \end{aligned}$$

Hence  $yz + zx + xy \leq (y + z - x)^2 + (z + x - y)^2 + (x + y - z)^2$ .

(b) —

(c) —

3. —

4. (a) **Answer.**

(I)  $n$

(II) an integer

(III)  $a_1, a_2, \dots, a_{n-1}, a_n$

(IV) positive

(V)  $\frac{a_1 + a_2 + \dots + a_{n-1} + a_n}{n}$

(VI) iff  $a_1 = a_2 = \dots = a_{n-1} = a_n$

(b) **Answer.**

(I)  $r_1, r_2, \dots, r_k$

(II) there exist

(III)  $M_1, M_2, \dots, M_k, N_1, N_2, \dots, N_k$

(IV)  $r_2 = \frac{M_2}{N_2}, \dots, r_k = \frac{M_k}{N_k}$

(V) positive

(VI)  $= M_1 + M_2 + \dots + M_k$

(VII)  $\frac{M_1}{N}x_1 + \frac{M_2}{N}x_2 + \dots + \frac{M_k}{N}x_k = \frac{M_1x_1 + M_2x_2 + \dots + M_kx_k}{N}$

(VIII)  $(x_1^{M_1}x_2^{M_2} \dots x_k^{M_k})^{1/N} = x_1^{M_1/N}x_2^{M_2/N} \dots x_k^{M_k/N}$

(c) —

5. (a) **Solution.**

Suppose  $w, x, y, z$  are real numbers. Then  $w^4, x^2y^2, y^2z^2, z^2x^2$  are all non-negative.

- (Case 1). Suppose some of  $w, x, y, z$  is zero. Then  $\frac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} \geq 0 = wxyz$ .
- (Case 2). Suppose none of  $w, x, y, z$  is zero. Then  $w^4, x^2y^2, y^2z^2, z^2x^2$  are all positive real numbers. By the Arithmetico-geometrical Inequality,

$$\begin{aligned} \frac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} &\geq (w^4 \cdot x^2y^2 \cdot y^2z^2 \cdot z^2x^2)^{\frac{1}{4}} \\ &= (w^4x^4y^4z^4)^{\frac{1}{4}} \\ &= |wxyz| \\ &\geq wxyz. \end{aligned}$$

Hence, in any case, we have  $\frac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} \geq wxyz$ .

(b) —

(c) —

6. (a) —

(b) i. **Answer.**

$R = 1, S = 1, T = 1$ .

ii. —

iii. —

iv. —

(c) —

(d) —

7. (a) **Answer.**

(I)  $h(x) \geq \sum_{j=0}^n \frac{x^{2j}}{j!}$  for any  $x \in [0, +\infty)$

$$(II) \ 1 + \int_0^x 0 du = 1 + 0 = 1$$

(III) Suppose  $P(k)$  is true

$$(IV) \ 1 + \int_0^x 2uh(u)du$$

$$(V) \ 1 + \int_0^x 2u \sum_{j=0}^k \frac{u^{2j}}{j!} du = 1 + \sum_{j=0}^k \int_0^x \frac{2u^{2j+1}}{j!} du = 1 + \sum_{j=0}^k \frac{u^{2j+2}}{(j+1)!} \Big|_0^x = 1 + \sum_{j=0}^k \frac{x^{2j+2}}{(j+1)!} = \sum_{j=0}^{k+1} \frac{x^{2j}}{j!}$$

(VI)  $P(k+1)$  is true

(b) —

8. —

9. —