MATH1050/1058 Assignment 7 (Answers and selected solutions)

1. **Solution.**

   (a) Take $n = 3$. Note that $3 \in \mathbb{N}$. Note that $3 + 2 = 5$, $3 + 4 = 7$. The integers $3, 5, 7$ are prime numbers.

   (b) Take $x = \sqrt{2}$. Note that $x \in \mathbb{R}$. We have $x^2 - 2 = (\sqrt{2})^2 - 2 = 2 - 2 = 0$.

   (c) Take $z_0 = \dfrac{1+i}{\sqrt{2}}$. Note that $z_0 \in \mathbb{C}$.

   Also note that $z_0{}^4 = \left(\dfrac{1+i}{\sqrt{2}}\right)^4 = \dfrac{1 + 4i + 6i^2 + 4i^3 + i^4}{4} = \dfrac{1 + 4i - 6 - 4i + 1}{4} = 1$.

   (d) Take $x = -\dfrac{1}{2}$. By definition, $x \in \mathbb{Q}$.

   Note that $(\log_2(-2x))^2 = \left(\log_2\left(-2 \cdot \left(-\frac{1}{2}\right)\right)\right)^2 = (\log_2(1))^2 = 0^2 = 0$.

   Also note that $-\log_2(4x^2) = -\log_2\left(4 \cdot \left(-\frac{1}{2}\right)^2\right) = -\log_2(1) = 0$.

   Then $(\log_2(-2x))^2 = -\log_2(4x^2)$.

2. **Answer.**

   (a) *There are many correct answers for (II), (III), ..., (IX) collectively, dependent on the choices made in (II).*

   (I) There exist some $x, y, z \in \mathbb{Z}$ such that each of xy, xz is divisible by 4 and xyz is not divisible by 8.
   (II) $y = z = 1$
   (III) 4
   (IV) 4
   (V) $4 = 1 \cdot 4$ and $1 \in \mathbb{Z}$
   (VI) 4
   (VII) 4 were divisible by 8
   (VIII) $4 = 8k$
   (IX) $\dfrac{1}{2}$

   (b) (I) There exist some sets A, B, C such that $A \cap B \neq \emptyset$ and $A \cap B \subset C$ and $A \not\subset C$ and $B \not\subset C$.
   (II) $C = \{3\}$
   (III) $\emptyset$
   (IV) $A \cap B \subset C$
   (V) and $1 \notin C$
   (VI) $A \not\subset C$
   (VII) $2 \in B$ and $2 \notin C$
   (VIII) $B \not\subset C$

   (c) (I) There exist some $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$ and $|x^2 - 2x| < |y^2 - 2y|$ and $x^2 > y^2$.
   (II) $y = 1$
   (III) $x > 0$ and $y > 0$
   (IV) 0
   (V) $|y^2 - 2y| = 1$
   (VI) $|x^2 - x|$
   (VII) $|y^2 - y|$
   (VIII) $x^2 = 4$
   (IX) $x^2 > y^2$

   (d) (I) There exist some $m, n \in \mathbb{N}\backslash\{0, 1, 2\}$, $\zeta, \omega \in \mathbb{C}$ such that $m \neq n$ and $\zeta \neq \omega$ and ζ is an m-th root of unity and ω is an n-th root of unity and $\zeta\omega$ is not an $(m+n)$-th root of unity.

(II) Take

(III) $\omega = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

(IV) $m \neq n$ and $\zeta \neq \omega$

(V) ζ is an m-th root of unity

(VI) $\omega^n = \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^8 = \cos\left(8 \cdot \frac{\pi}{4}\right) + i\sin\left(8 \cdot \frac{\pi}{4}\right) = \cos(2\pi) + i\sin(2\pi) = 1$

(VII) 12

(VIII) $(\zeta\omega)^{m+n} = \left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^{12} = \cos\left(12 \cdot \frac{3\pi}{4}\right) + i\sin\left(12 \cdot \frac{3\pi}{4}\right) = \cos(9\pi) + i\sin(9\pi) = -1$

(IX) $\neq$

(X) $\zeta\omega$ is not an $(m+n)$-th root of unity

3. **Solution.**

(a) Let $z \in \mathbb{C}\backslash\{0\}$. Suppose it were true that $\mathsf{Re}(z) = 0$ and $\mathsf{Im}(z) = 0$. Then $z = \mathsf{Re}(z) + i\mathsf{Im}(z) = 0 + i \cdot 0 = 0$. Contradiction arises. Hence $\mathsf{Re}(z) \neq 0$ or $\mathsf{Im}(z) \neq 0$ in the first place.

(b) The statement 'for any $z \in \mathbb{C}\backslash\{0\}$, $\mathsf{Re}(z) \neq 0$' is false: we have $i \in \mathbb{C}\backslash\{0\}$ and $\mathsf{Re}(i) = 0$. The statement 'for any $w \in \mathbb{C}\backslash\{0\}$, $\mathsf{Im}(w) \neq 0$' is also false: we have $1 \in \mathbb{C}$ and $\mathsf{Im}(1) = 0$.

Hence the statement '(for any $z \in \mathbb{C}\backslash\{0\}$, $\mathsf{Re}(z) \neq 0$) or (for any $w \in \mathbb{C}\backslash\{0\}$, $\mathsf{Im}(w) \neq 0$)' is false.

4. **Solution.**

(a) i. $-2 \in \mathbb{Z}$. $-2 + 1 = -1 < 0$.

ii. $2 \in \mathbb{Z}$. $2 - 1 = 1 > 0$.

(b) Suppose it were true that there existed some $x \in \mathbb{Z}$ such that ($x + 1 < 0$ and $x - 1 > 0$). For this x, we would have $x < -1$ and $x > 1$. Then $x < -1 < 1 < x$. Therefore $x \neq x$. Contradiction arises.

Hence it is false that there exists some $x \in \mathbb{Z}$ such that ($x + 1 < 0$ and $x - 1 > 0$).

Alterntative argument: The negation of the statement 'there exists some $x \in \mathbb{Z}$ such that ($x+1 < 0$ and $x-1 > 0$)' is given by:

- For any $x \in \mathbb{Z}$, ($x + 1 \geq 0$ or $x - 1 \leq 0$).

We give a proof of the latter:

- Let $x \in \mathbb{Z}$. We have $x \geq 0$ or $x \leq 0$. Where $x \geq 0$, we have $x+1 \geq 1 \geq 0$. Where $x \leq 0$, we have $x-1 \leq -1 \leq 0$.

5. **Answer.**

(a) (I) Suppose

(II) $u \in \mathbb{R}\backslash\{-1, 0, 1\}$

(III) $u^6 + v^6 \leq 2v^4$

(IV) $u^6 - 2u^4 + u^2 + v^6 - 2v^4 + v^2 \leq 0$

(V) $u^2(u^2 - 1)^2 = 0$

(VI) $u \in \mathbb{R}\backslash\{-1, 0, 1\}$

(b) (I) Suppose there existed some $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that ζ was both an 89-th root of unity and a 55-th root of unity.

(II) 1

(III) $\zeta^{89} = 1$

(IV) $\zeta^{21} = \zeta^{55}/\zeta^{34} = 1$, $\zeta^{13} = \zeta^{34}/\zeta^{21} = 1$, $\zeta^8 = \zeta^{21}/\zeta^{13} = 1$, $\zeta^5 = \zeta^{13}/\zeta^8 = 1$, $\zeta^3 = \zeta^8/\zeta^5 = 1$, $\zeta^2 = \zeta^5/\zeta^3 = 1$, $\zeta = \zeta^3/\zeta^2 = 1$.

(V) $\mathbb{C}\backslash\mathbb{R}$

(VI) and

6. **Answer.**

(a) Least element: -1.

Greatest element: *None*.

The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is an upper bound.)

(b) Least element: *None*.

The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)

Greatest element: *None*.

The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 ia an upper bound.)

(c) Least element: *None*.

The set concerned is bounded below by 0 in $\mathbb{R}$. (Every real number no greater than 0 is a lower bound.)

Greatest element: 1.

(d) Least element: *None*.

The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)

Greatest element: 2.

(e) Least element: *None*.

The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no greater than 1 is a lower bound.)

Greatest element: *None*.

The set concerned is not bounded above in $\mathbb{R}$.

(f) Least element: *None*.

The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no greater than 1 is a lower bound.)

Greatest element: *None*.

The set concerned is not bounded above in $\mathbb{R}$.

(g) Least element: *None*.

The set concerned is bounded below by $-\frac{3}{2}$ in $\mathbb{R}$. (Every real number no greater than $-\frac{3}{2}$ is a lower bound.)

Greatest element: *None*

The set concerned is not bounded above in $\mathbb{R}$.

(h) Least element: -1.

Greatest element: 2.

(i) Least element: *None*.

The set concerned is not bounded below in $\mathbb{R}$.

Greatest element: *None*.

The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

(j) Least element: *None*.

The set concerned is bounded below by -3 in $\mathbb{R}$. (Every real number no greater than -3 is a lower bound.)

Greatest element: *None*.

The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

(k) Least element: *None*.

The set concerned is bounded below by 0 in $\mathbb{R}$. (Every real number no greater than 0 is a lower bound.)

Greatest element: 2.

(l) Least element: *None*.

The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)

Greatest element: *None*.

The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

7. **Answer.**

(a) (I) $\frac{1}{\sqrt{2}}$

(II) $\lambda = 0 \cdot 1 + \frac{1}{2} \cdot \sqrt{2}$

(III) $\mathbb{Q}$

(IV) $\frac{1}{\sqrt{2}} \leq \lambda < \sqrt{2}$

(V) $\lambda \in B$

(VI) and

(VII) Pick any $x \in C$

(VIII) and $x \in B$

(IX) $\frac{1}{\sqrt{2}} \leq x < \sqrt{2}$

(X) $x \geq \lambda$

(b) (I) Suppose

(II) a greatest element in $\mathbb{R}$

(III) $\mu \in A$ and $\mu \in B$

(IV) there would exist some $a, b \in \mathbb{Q}$ such that

(V) $\mu \in B$

(VI) $\frac{1}{\sqrt{2}} \leq \mu < x_0 < \sqrt{2}$

(VII) $\frac{a}{2} + \frac{b+1}{2}\sqrt{2}$

(VIII) $a \in \mathbb{Q}$

(IX) $\frac{b+1}{2} \in \mathbb{Q}$

(X) $x_0 \in A$

(XI) $x_0 \in C$

(XII) μ was a greatest element of C

8. **Answer.**

(a) ——

(b) $\frac{1}{25}$ is the least element of T.

(c) *Hint.* $\frac{1}{125}$ is an element of S and is not an element of T.

(d) *Hint.* Given that $u, v \in S$ and $u < v$, is it true that $\frac{4u+v}{5} \in S$ and $u < \frac{4u+v}{5} < v$?

9. **Answer.**

(a) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence.

(b) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence.

(c) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence.

(d) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence.

(e) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence

(f) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence.

(g) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence.

(h) This infinite sequence is strictly increasing.
 3/2 is an upper bound of this infinite sequence.

(i) This infinite sequence is strictly increasing.
 3/2 is an upper bound of this infinite sequence.

(j) This infinite sequence is strictly decreasing.
 0 is a lower bound for this infinite sequence.

(k) This infinite sequence is strictly increasing.
 1 is an upper bound of this infinite sequence.

(l) This infinite sequence is strictly decreasing.
 0 is a lower bound of this infinite sequence.

(m) This infinite sequence is strictly increasing.
 1 is an upper bound of this infinite sequence.

(n) This infinite sequence is strictly increasing.
 1 is an upper bound of this infinite sequence.

10. ——

11. (a) ——

 (b) **Answer.**

 $\lim_{n\to\infty} a_n = \alpha.$