1. *This is a review question on the framework of proofs of statements starting with* there exists.

   Prove each of the 'existence statements' below. (The proofs are easy: conceive the candidates and verify the candidacty. Do not think too hard.)

   (a) *There exists some $n \in \mathbb{N}$ such that $n, n+2, n+4$ are prime numbers.*

   (b) *There exists some $x \in \mathbb{R}$ such that $x^2 - 2 = 0$.*

   (c) *There exists some $z \in \mathbb{C}$ such that $z^4 = -1$.*

   (d) *There exists some $x \in \mathbb{Q}$ such that $(\log_2(-2x))^2 = -\log_2(4x^2)$.*

2. Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a dis-proof against the statement (A), a dis-proof against the statement (B), a dis-proof against the statement (C) and a dis-proof against the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) We dis-prove the statement (A):

   (A) *Let $x, y, z \in \mathbb{Z}$. Suppose each of xy, xz is divisible by 4. Then xyz is divisible by 8.*

   > [The negation of the statement (A) is given by:
   >
   > $(\sim A)$ __________ (I) __________]
   >
   > - We verify the negation of the statement (A) below:
   >
   >   Take $x = 4$, ____ (II) ____ . We have $x, y, z \in \mathbb{Z}$.
   >
   >   Note that $xy =$ __ (III) __ , and $xz =$ __ (IV) __ .
   >
   >   Note that ____ (V) ____ . Then xy is divisible by 4.
   >
   >   By a similar argument, we also deduce that xz is divisible by 4.
   >
   >   Note that $xyz =$ __ (VI) __ , which is not divisible by 8.
   >
   >   Below is the justification of this claim:
   >
   >   * Suppose it were true that ____ (VII) ____ .
   >
   >   Then there would exist some $k \in \mathbb{Z}$ such that ____ (VIII) ____ .
   >
   >   For the same k, we would have $k =$ __ (IX) __ , which is not an integer. Contradiction arises.

   (b) We dis-prove the statement (B):

   (B) *Let A, B, C be sets. Suppose $A \cap B \neq \emptyset$ and $A \cap B \subset C$. Then $A \subset C$ or $B \subset C$.*

   > [The negation of the statement (B) is given by:
   >
   > $(\sim B)$ __________ (I) __________]
   >
   > We verify the negation of the statement (B) below:
   >
   > - Take $A = \{1, 3\}$, $B = \{2, 3\}$ and ____ (II) ____ .
   >
   >   We have $A \cap B = \{3\}$. Then $A \cap B \neq$ __ (III) __ .
   >
   >   Moreover $A \cap B = C$. Then ____ (IV) ____ .
   >
   >   We verify $A \not\subset C$ and $B \not\subset C$:
   >
   >   * We have $1 \in A$ ____ (V) ____ . Then ____ (VI) ____ .
   >
   >   We have ____ (VII) ____ . Then ____ (VIII) ____ .
   >
   >   Hence $A \not\subset C$ and $B \not\subset C$ (simultaneously).

(c) We dis-prove the statement (C):

(C) *Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$ and $|x^2 - 2x| < |y^2 - 2y|$. Then $x^2 \leq y^2$.*

> [The negation of the statement (C) is given by:
>
> $(\sim C)$ ______(I)______]
>
> We verify the negation of the statement (C) below:
>
> - Take $x = 2$, ___(II)___ . We have $x, y \in \mathbb{R}$, and ___(III)___ .
>
>   Note that $|x^2 - 2x| =$ ___(IV)___ and ___(V)___ . Then ___(VI)___ $<$ ___(VII)___ .
>
>   We have ___(VIII)___ and $y^2 = 1$. Then ___(IX)___ .

(d) We dis-prove the statement (D):

(D) *Let $m, n \in \mathbb{N}\backslash\{0, 1, 2\}$ and $\zeta, \omega \in \mathbb{C}$. Suppose $m \neq n$, $\zeta \neq \omega$, ζ is an m-th root of unity and ω is an n-th root of unity. Then $\zeta\omega$ is an $(m + n)$-th root of unity.*

> [The negation of the statement (D) is given by:
>
> $(\sim D)$ ______(I)______]
>
> We verify the negation of the statement (D) below:
>
> - ___(II)___ $m = 4$, $n = 8$, $\zeta = i$ and ___(III)___ .
>
>   We have $m, n \in \mathbb{N}\backslash\{0, 1, 2\}$ and $\zeta, \omega \in \mathbb{C}$. Also, ___(IV)___ .
>
>   Note that $\zeta^m = i^4 = 1$. Then ___(V)___ .
>
>   Note that ___(VI)___ . Then ω is an n-th root of unity.
>
>   Now note that $m + n =$ ___(VII)___ and $\zeta\omega = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$.
>
>   We have ___(VIII)___ . Then $(\zeta\omega)^{m+n}$ ___(IX)___ 1.
>
>   Therefore ___(X)___ .

3. (a) Prove the statement $(\sharp)$:

   $(\sharp)$ *For any $z \in \mathbb{C}\backslash\{0\}$, $(\mathsf{Re}(z) \neq 0$ or $\mathsf{Im}(z) \neq 0)$.*

   (b) Dis-prove the statement $(\flat)$:

   $(\flat)$ *(For any $z \in \mathbb{C}\backslash\{0\}$, $\mathsf{Re}(z) \neq 0$) or (for any $w \in \mathbb{C}\backslash\{0\}$, $\mathsf{Im}(w) \neq 0$).*

Remark. It can happen that $(\forall x)[H(x) \to (P(x) \vee Q(x))]$ is true and $[(\forall x)(H(x) \to P(x))] \vee [(\forall y)(H(y) \to Q(y))]$ is false. In general, $(\forall x)[H(x) \to (P(x) \vee Q(x))]$ does not imply $[(\forall x)(H(x) \to P(x))] \vee [(\forall y)(H(y) \to Q(y))]$.

4. (a) Prove each of the statements below:

   i. *There exists some $x \in \mathbb{Z}$ such that $x + 1 < 0$.*
   ii. *There exists some $x \in \mathbb{Z}$ such that $x - 1 > 0$.*

   (b) Dis-prove the statement $(\natural)$:

   $(\natural)$ *There exists some $x \in \mathbb{Z}$ such that $(x + 1 < 0$ and $x - 1 > 0)$.*

Remark. It can happen that $[(\exists x)P(x)] \wedge [(\exists y)Q(y)]$ is true while $(\exists x)(P(x) \wedge Q(x))$ is false. In general, $[(\exists x)P(x)] \wedge [(\exists y)Q(y)]$ does not imply $(\exists x)(P(x) \wedge Q(x))$.

5. Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a dis-proof against the statement (D) and a dis-proof for the statement (E). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) We dis-prove the statement (D):

   (D) *There exist some $u \in \mathbb{R}\backslash\{-1, 0, 1\}$, $v \in \mathbb{R}$ such that $u^2 + v^2 \leq 2u^4$ and $u^6 + v^6 \leq 2v^4$.*

[We dis-prove the statement (D) by obtaining a contradiction from it.]

___(I)___ there existed some ___(II)___ , $v \in \mathbb{R}$ such that $u^2 + v^2 \leq 2u^4$ and ___(III)___ .

For the same u, v, we would have $u^2 + v^2 - 2u^4 \leq 0$ and $u^6 + v^6 - 2v^4 \leq 0$.

Then $u^2(u^2-1)^2 + v^2(v^2-1)^2 =$ ___(IV)___ .

Since u, v are real, $u^2(u^2-1)^2 \geq 0$ and $v^2(v^2-1)^2 \geq 0$. Then $u^2(u^2-1)^2 = 0$ and $v^2(v^2-1)^2 = 0$ respectively.

In particular, ___(V)___ . Then $u = 0$ or $u = -1$ or $u = 1$. But ___(VI)___ .

Contradiction arises.

(b) We dis-prove the statement (E):

(E) There exist some $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that ζ is both an 89-th root of unity and a 55-th root of unity.

[We dis-prove the statement (E) by obtaining a contradiction from it.]

___(I)___

For the same ζ, we would have $\zeta^{55} =$ ___(II)___ and ___(III)___ by the definition of root of unity.

(Note that $\zeta \neq 0$.) Then we would have $\zeta^{34} = \zeta^{89-55} = \zeta^{89}/\zeta^{55} = 1$.

Repeating the above argument, we would have:

___(IV)___

Recall that by assumption, $\zeta \in$ ___(V)___ . Then $\zeta \neq 1$.

Now $\zeta = 1$ ___(VI)___ $\zeta \neq 1$.

Contradiction arises.

6. Consider each of the subsets of $\mathbb{R}$ below.

- Determine whether it has any least element. If *yes*, name it as well. If it has no least element, determine whether it has a lower bound in $\mathbb{R}$.
- Determine whether it has any greatest element. If *yes*, name it as well. If it has no greatest element, determine whether it has an upper bound in $\mathbb{R}$.

There is no need to justify your answers. (Drawing appropriate pictures, on the real line or on the coordinate plane, may help you find the answers.)

(a) $[-1, 1) \cap \mathbb{Q}$

(b) $[-1, 1)\backslash\mathbb{Q}$

(c) $\left\{ \dfrac{1}{n+1} \,\middle|\, n \in \mathbb{N}. \right\}$

(d)$\diamondsuit$ $\left\{ \dfrac{1}{n+1} + (-1)^n \,\middle|\, n \in \mathbb{N}. \right\}$

(e) $(1, +\infty) \cap \mathbb{Q}$

(f) $(1, +\infty)\backslash\mathbb{Q}$

(g) $\{x \in \mathbb{R} : 2x + 3 > 0.\}$

(h) $\{x \in \mathbb{R} : x + 2 \geq x^2.\}$

(i) $\{x \in \mathbb{R} : x < x^{-1}.\}$

(j) $\{x \in \mathbb{R} : x^2 - 2x - 3 < 0.\}$

(k)$\diamondsuit$ $\left\{ \dfrac{1}{2^m} + \dfrac{1}{3^n} \,\middle|\, m, n \in \mathbb{N}. \right\}$

(l)$\clubsuit$ $\left\{ \dfrac{1}{2^m} - \dfrac{1}{3^n} \,\middle|\, m, n \in \mathbb{N}. \right\}$

7. Let $A = \{x \in \mathbb{R} : x = a + b\sqrt{2} \text{ for some } a, b \in \mathbb{Q}\}$, $B = \left[\dfrac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $C = A \cap B$.

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they gives a proof for the statement (P) and a dis-proof against the statement (Q).

(a) Here we prove the statement (P):

(P) *C has a least element.*

Take $\lambda =$ ___(I)___ .

- We have ___(II)___ , and $0, \dfrac{1}{2} \in$ ___(III)___ . Then by the definition of A, we have $\lambda \in A$.

  Note that ___(IV)___ . Then by the definition of B, we have ___(V)___ .

  Now we have $\lambda \in A$ ___(VI)___ $\lambda \in B$. Therefore, by the definition of C, we have $\lambda \in C$.

- ___(VII)___ . Then by the definition of C, we have $x \in A$ ___(VIII)___ . Therefore $x \in B$ in particular.

  Then by the definition of B, we have ___(IX)___ .

  Recall that $\lambda = \dfrac{1}{\sqrt{2}}$. Therefore ___(X)___ .

It follows that λ is a least element of C.

(b) Here we dis-prove the statement (Q):

(Q) *C has a greatest element.*

> ___(I)___ it were true that C had ___(II)___, which we denote by μ.
> Then, by definition, $\mu \in C$. Therefore ___(III)___ by the definition of C.
> Since $\mu \in A$, ___(IV)___ $\mu = a + b\sqrt{2}$.
> Since ___(V)___, we would have $\frac{1}{\sqrt{2}} \leq \mu < \sqrt{2}$.
> Define $x_0 = \frac{\mu + \sqrt{2}}{2}$. By definition, we would have ___(VI)___. Then $x_0 > \mu$ and $x_0 \in B$.
> Also by definition, $x_0 = \frac{\mu + \sqrt{2}}{2} = \frac{a + b\sqrt{2} + \sqrt{2}}{2} =$ ___(VII)___.
> Since ___(VIII)___, we would have $\frac{a}{2} \in \mathbb{Q}$. Since $b \in \mathbb{Q}$, we would have ___(IX)___. Then ___(X)___.
> Now $x_0 \in A$ and $x_0 \in B$. Then ___(XI)___ by definition.
> But $x_0 > \mu$, and ___(XII)___. Contradiction arises.

8. Let $S = \left\{ x \in \left(0, \frac{1}{24}\right) : x = \frac{b}{5^a} \text{ for some } a, b \in \mathbb{N} \right\}$, and $T = \left\{ y \in \mathbb{R} : y = \sum_{k=1}^{n} \frac{1}{25^k} \text{ for some } n \in \mathbb{N}\backslash\{0\} \right\}$.

   (a) Verify that $T \subset S$.

   (b) Does T have a least element? Justify your answer.

   (c) Prove that $S \not\subset T$.

   Remark. The result you obtain in part (b) may be useful.

   (d)$^\diamondsuit$ Prove the statement ($\sharp$):

   ($\sharp$) *For any $u, v \in S$, if $u < v$ then there exists some $w \in S$ such that $u < w < v$.*

9. Consider each of the infinite sequences (of non-negative real numbers) below. Determine whether it is strictly increasing or strictly decreasing or neither. Where it is strictly increasing/decreasing, determine whether it is bounded above/below in $\mathbb{R}$; if it is bounded above/below in $\mathbb{R}$, name an upper/lower bound for it.

   There is no need to justify your answers.

   (a) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

   (b) $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$

   (c) $\left\{\frac{n+1}{n^2+n+1}\right\}_{n=0}^{\infty}$

   (d) $\left\{\frac{2}{3^n}\right\}_{n=0}^{\infty}$

   (e) $\left\{\frac{5^n}{n!}\right\}_{n=5}^{\infty}$

   (f) $\left\{\frac{(n!)^2}{(2n)!}\right\}_{n=0}^{\infty}$

   (g) $\left\{\sqrt[n]{2}\right\}_{n=2}^{\infty}$

   (h) $\left\{\sum_{k=0}^{n}\frac{1}{3^k}\right\}_{n=0}^{\infty}$

   (i)$^\diamondsuit$ $\left\{\sum_{k=2}^{n}\frac{2}{k^2-1}\right\}_{n=2}^{\infty}$

   (j) $\left\{\frac{n^2}{3^n}\right\}_{n=2}^{\infty}$

   (k) $\left\{\sqrt[n]{\frac{2}{3}}\right\}_{n=2}^{\infty}$

   (l) $\left\{1-\prod_{k=2}^{n}\cos\left(\frac{\pi}{2^k}\right)\right\}_{n=2}^{\infty}$

   (m) $\left\{\prod_{k=1}^{n}\frac{1}{2^k}\right\}_{n=1}^{\infty}$

   (n) $\left\{1-\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right\}_{n=1}^{\infty}$

10. Define $a_n = \sum_{k=1}^{n}\frac{1}{k^2}$, $b_n = \sum_{k=1}^{n}\frac{1}{k^3}$ for each $n \geq 1$.

    (a)$^\clubsuit$ Prove that $a_{2^{m+1}-1} - a_{2^m-1} \leq \frac{1}{2^m}$ for each $m \geq 1$. Hence deduce that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

    (*Hint.* It help to observe that $a_n \leq a_{2^n-1}$ for each n. But is this observation true? Why?)

    (b)$^\diamondsuit$ By applying the result in the previous part, or otherwise, prove that $\{b_n\}_{n=1}^{\infty}$ is bounded above by 2.

11. Let p be a positive real number, and $\alpha = \sqrt[3]{p}$. Suppose $b \in (\alpha, +\infty)$. Let $\{a_n\}_{n=0}^{\infty}$ be the infinite sequence defiend by

$$\begin{cases} a_0 &= b \\ a_{n+1} &= \frac{1}{3}\left(2a_n + \frac{\alpha^3}{{a_n}^2}\right) \quad \text{for any } n \in \mathbb{N} \end{cases}$$

   (a)$^\clubsuit$ Prove the statements below:

   i. *$\{a_n\}_{n=0}^{\infty}$ is bounded below by α in $\mathbb{R}$.*

   ii. *$\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.*

   (b) Apply the Bounded-Monotone Theorem to prove that $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$. Also find the value of $\lim_{n\to\infty} a_n$.