MATH1050/1058 Assignment 6 (Answers and selected solutions)

1. (a) (I) $r = a + b\sqrt{2}$ and $r = a' + b'\sqrt{2}$

(II) $a = a'$ and $b = b'$

(III) and $r = a' + b'\sqrt{2}$

(IV) $(b - b')\sqrt{2}$

(V) Suppose it were true that $b \neq b'$

(VI) $\sqrt{2} = \dfrac{a' - a}{b - b'}$

(VII) $\sqrt{2}$ would be a rational number

(VIII) $b = b'$

(b) (I) $\zeta \in \mathbb{C}\backslash\mathbb{R}$ and $\eta \in \mathbb{C}$

(II) For any $a, a', b, b' \in \mathbb{R}$, if $\eta = a\zeta + b\zeta^2$ and $\eta = a'\zeta + b'\zeta^2$ then $a = a'$ and $b = b'$.

(III) Pick any $a, a', b, b' \in \mathbb{R}$.

(IV) Suppose

(V) $\eta = a'\zeta + b'\zeta^2$

(VI) $\zeta \neq 0$

(VII) $a' - a$

(VIII) Suppose it were true that $b \neq b'$

(IX) $\dfrac{a' - a}{b - b'}$

(X) real

(XI) ζ is not real

(XII) $a' = a$

(c) (I) Suppose $r \in \mathbb{R}$.

(II) For any $n, n' \in \mathbb{Z}$, if $n \leq r < n + 1$ and $n' \leq r < n' + 1$ then $n = n'$.

(III) Pick any $n, n' \in \mathbb{Z}$. Suppose $n \leq r < n + 1$ and $n' \leq r < n' + 1$.

(IV) $(r - n') - (r - n) < 1 - 0$

(V) $(r - n') - (r - n) > 0 - 1$

(VI) n, n' are integers

(VII) an integer

(VIII) only

(IX) 0

(X) $n - n' = 0$

2. (a) **Answer.**

(I) Suppose

(II) Suppose s is not divisible by 2.

(III) there exist some $k, r \in \mathbb{Z}$ such that $s = 2k + r$ and $0 \leq r < 2$

(IV) s is not divisible by 2

(V) $0 < r < 2$

(VI) $r \in \mathbb{Z}$

(VII) $s = 2k + 1$

(VIII) if there exists some $k \in \mathbb{Z}$ such that $s = 2k + 1$ then s is not divisible by 2

(IX) Suppose it were true that s was divisible by 2.

(X) there would exist some $\ell \in \mathbb{Z}$ such that $s = 2\ell$

(XI) $s = 2k + 1$ and $s = 2\ell + 0$

(XII) By the Division Algorithm for Integers

(XIII) $0 = 1$

(b) ——

(c) ——

3. ——

4. **Solution.**

   Let n be a positive integer.

   Since n is a positive integer, we have $n^7 + n^6 + n^5 + n^4 + n^3 + n^2 + n + 1 > n^4 + n^3 + n^2 + n + 1 > 0$.

   Repeatedly applying Division Algorithm, we obtain:

$$\begin{cases} n^7 + n^6 + n^5 + n^4 + n^3 + n^2 + n + 1 & = & n^3(n^4 + n^3 + n^2 + n + 1) & + & (n^2 + n + 1) \\ n^4 + n^3 + n^2 + n + 1 & = & n^2(n^2 + n + 1) & + & (n + 1) \\ n^2 + n + 1 & = & n(n + 1) & + & 1 \end{cases}$$

   Since n is a positive integer, we indeed have the inequalities $n^4 + n^3 + n^2 + n + 1 > n^2 + n + 1 > n + 1 > 1 > 0$.

   Hence the greatest common divisor of $n^7 + n^6 + n^5 + n^4 + n^3 + n^2 + n + 1$ and $n^4 + n^3 + n^2 + n + 1$ is 1.

5. ——

6. (a) **Answer.**

   (I) Suppose a, c are relatively prime and ab is divisible by c

   (II) ab is divisible by c

   (III) $k \in \mathbb{Z}$

   (IV) $\gcd(a, c) = 1$

   (V) there exist some $s, t \in \mathbb{Z}$

   (VI) $sa + tc$

   (VII) $\gcd(a, c)$

   (VIII) $(sa + tc)b = sab + tbc = skc + tbc = (sk + tb)c$

   (IX) $sk + tb$

   (X) b is divisible by c

   (b) i. ——

   ii. *Hint.*
   Apply the result in part (b.i).

   iii. ——

   iv. *Hint.*
   Apply the result in part (b.iii).

   v. ——

7. (a) (I) $A \cup B \subset B$

   (II) it were true that $A \backslash B \neq \emptyset$

   (III) $x_0 \in A \backslash B$

   (IV) $x_0 \in A$

   (V) $x_0 \in B$

   (VI) $x_0 \in A \cup B$

   (VII) $x_0 \in B$

   (VIII) Contradiction arises

   (b) (I) Suppose $S \in \mathfrak{P}(C) \cup \mathfrak{P}(D)$

   (II) $S \in \mathfrak{P}(C)$ or $S \in \mathfrak{P}(D)$

   (III) Suppose $S \in \mathfrak{P}(C)$.

   (IV) $S \subset C$

   (V) Since $x \in S$ and $S \subset C$, we have $x \in C$

   (VI) $x \in C$ or $x \in D$

(VII) $x \in C \cup D$

(VIII) $S \subset C \cup D$

(IX) $S \in \mathfrak{P}(C \cup D)$

(X) $S \in \mathfrak{P}(C \cup D)$

(c) (I) if $x \in A \cap B$ then $x \in A$

(II) Suppose $x \in A \cap B$

(III) $x \in A$ and $x \in B$

(IV) $x \in A$

(V) For any object x, if $x \in A$ then $x \in A \cap B$

(VI) Pick any object x. Suppose $x \in A$.

(VII) $x \in A$ and $A \subset B$

(VIII) $x \in A$ and $x \in B$

(IX) $x \in A \cap B$

(X) $A \cap B \subset A$

(XI) $A \cap B = A$

(XII) For any object x, if $x \in A$ then $x \in B$

(XIII) $x \in A$

(XIV) $A \cap B = A$

(XV) by the definition of intersection, we have $x \in A$ and $x \in B$

(XVI) $x \in B$

(d) (I) Suppose $x \in C \backslash B$.

(II) complement

(III) $x \in C$ and $x \notin B$

(IV) Suppose it were true that $x \in A$.

(V) since $x \in A$ and

(VI) $x \in B$

(VII) and

(VIII) $x \in C$ and $x \notin A$

(IX) $x \in C \backslash A$

(X) Pick any object x. Suppose $x \in A$.

(XI) Suppose it were true that $x \notin B$.

(XII) and $A \subset C$

(XIII) $x \in C$

(XIV) $x \in C$ and $x \notin B$

(XV) $x \in C \backslash B$

(XVI) $x \in C \backslash A$

(XVII) complement

(XVIII) $x \in C$ and $x \notin A$

(XIX) $x \notin A$

(XX) $x \in A$ and $x \notin A$