MATH1050/1058 Assignment 4 (Answers and selected solutions)

1. **Solution.**

   Let $\zeta = \sin(\frac{2\pi}{3}) + i\cos(\frac{2\pi}{3})$.

   (a) $\zeta = \cos(\frac{\pi}{2} - \frac{2\pi}{3}) + i\sin(\frac{\pi}{2} - \frac{2\pi}{3}) = \cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})$.

   (b) The three cubic roots of ζ are given by β_1, β_2, β_3, given by

$$\begin{aligned}
\beta_1 &= (\cos(-\frac{\pi}{18}) + i\sin(-\frac{\pi}{18})), \\
\beta_2 &= (\cos(-\frac{\pi}{18} + \frac{2\pi}{3}) + i\sin(-\frac{\pi}{18} + \frac{2\pi}{3}) = (\cos(\frac{11\pi}{18}) + i\sin(\frac{11\pi}{18})), \\
\beta_3 &= (\cos(-\frac{\pi}{18} - \frac{2\pi}{3}) + i\sin(-\frac{\pi}{18} - \frac{2\pi}{3}) = (\cos(-\frac{13\pi}{18}) + i\sin(-\frac{13\pi}{18})).
\end{aligned}$$

2. **Answer.**

   (a) z is a solution of the equation concerned iff $z = 2(\cos(\frac{\pi}{2} + N\cdot\frac{2\pi}{5}) + i\sin(\frac{\pi}{2} + N\cdot\frac{2\pi}{5}))$ for some N amongst $0, 1, 2, 3, 4$.

   (b) $z = 2(\cos(\frac{\pi}{10}) + i\sin(\frac{\pi}{10})$ or $z = 2(\cos(-\frac{3\pi}{10}) + i\sin(-\frac{3\pi}{10})$ or $z = 2(\cos(-\frac{7\pi}{10}) + i\sin(-\frac{7\pi}{10}))$.

3. **Answer.**

   (a) The quintic roots of ω are given by:

$$\begin{aligned}
\zeta_1 &= \cos\left(\frac{2\pi}{15}\right) + i\sin\left(\frac{2\pi}{15}\right), \\
\zeta_2 &= \cos\left(\frac{8\pi}{15}\right) + i\sin\left(\frac{8\pi}{15}\right), \\
\zeta_3 &= \cos\left(\frac{14\pi}{15}\right) + i\sin\left(\frac{14\pi}{15}\right), \\
\zeta_4 &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right), \\
\zeta_5 &= \cos\left(\frac{26\pi}{15}\right) + i\sin\left(\frac{26\pi}{15}\right).
\end{aligned}$$

   (b) $f(z) = \left(z^2 - 2z\cos\left(\frac{2\pi}{15}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{8\pi}{15}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{14\pi}{15}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{3}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{26\pi}{15}\right) + 1\right)$

   (c) i. 0.

   ii. $-\frac{5}{4}$.

4. **Answer.**

   (a) (I) '$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$'

   (II) $P(1)$ is true

   (III) $P(k)$ is true

   (IV) 0

(V)

$$\begin{aligned}
& (2\sqrt{k+1}-1) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right) \\
= \;& 2(\sqrt{k+1}-\sqrt{k}) - \frac{1}{\sqrt{k+1}} + (2\sqrt{k}-1) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}}\right) \\
\geq \;& 2(\sqrt{k+1}-\sqrt{k}) - \frac{1}{\sqrt{k+1}} \\
= \;& \frac{2}{\sqrt{k+1}+\sqrt{k}} - \frac{1}{\sqrt{k+1}} \\
= \;& \frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k+1}+\sqrt{k})\sqrt{k+1}}
\end{aligned}$$

(VI) $2\sqrt{k+1}-1$

(VII) $P(k+1)$ is true

(VIII) By the Principle of Mathematical Induction, $P(n)$ is true whenever n is a positive integer.

(b) (I) Denote by $P(n)$ the proposition '$n(n^2+2)$ is divisible by 3'.

(II) $0 \cdot (0^2+2) = 0 = 3 \cdot 0$ and $0 \in \mathbb{Z}$

(III) $0 \cdot (0^2+2)$ is divisible by 3

(IV) Let k be a natural number. Suppose $P(k)$ is true.

(V) $k(k^2+2)$

(VI) there exists some $q \in \mathbb{Z}$ such that $k(k^2+2) = 3q$

(VII) $(k+1)[(k+1)^2+2] = k^3+3k^2+5k+3 = k(k^2+2)+3k^2+3 = 3q+3k^2+3 = 3(q+k^2+1)$

(VIII) $q+k^2+1$

(IX) $(k+1)[(k+1)^2+2]$ is divisible by 3

(X) By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

5. **Answer.**

(a) (I) Suppose $\displaystyle\sum_{j=0}^{n} a_j = \left(\frac{1+a_n}{2}\right)^2$ for each $n \in \mathbb{N}$.

(II) '$a_n = 2n+1$'

(III) We have $a_0 = \displaystyle\sum_{j=0}^{0} a_j = \left(\frac{1+a_0}{2}\right)^2 = \frac{1}{4}(1+2a_0+{a_0}^2)$. Then $(a_0-1)^2 = {a_0}^2 - 2a_0 + 1 = 0$. Therefore $a_0 = 1 = 2 \cdot 0 + 1$.

(IV) Let $k \in \mathbb{N}$. Suppose $P(k)$ is true.

(V) We have

$$\left(\frac{1+a_{k+1}}{2}\right)^2 = \sum_{j=0}^{k+1} a_j = \sum_{j=0}^{k} a_j + a_{k+1} = \left(\frac{1+a_k}{2}\right)^2 + a_{k+1} = \left[\frac{1+(2k+1)}{2}\right]^2 + a_{k+1} = (k+1)^2 + a_{k+1}.$$

Then $\frac{1}{4}(1+2a_{k+1}+{a_{k+1}}^2) = (k+1)^2 + a_{k+1}$.

Therefore $(a_{k+1}-1)^2 = {a_{k+1}}^2 - 2a_{k+1} + 1 = (2k+2)^2$.

Hence $a_{k+1} = 2k+3$ or $a_{k+1} = -2k-1$. Since $a_{k+1} > 0$, we have $a_{k+1} = 2k+3 = 2(k+1)+1$.

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(b) (I) Let α, β are the two distinct roots of the polynomial $f(x) = x^2 - x - 1$. Suppose $\{a_n\}_{n=1}^{\infty}$ is the infinite sequence of real numbers defined by

$$\begin{cases} a_1 = 1, & a_2 = 3, \\ & a_{n+2} = a_{n+1} + a_n \quad \text{if} \quad n \geq 1 \end{cases}.$$

(II) $1 = -(-1) = \alpha + \beta$

(III) $3 = [-(-1)]^2 - 2(-1) = (\alpha+\beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$

(IV) Let k be a positive integer. Suppose $P(k)$ is true.

(V) $\alpha^{k+1} + \beta^{k+1}$

(VI) $P(k)$

(VII) $a_{k+2} = a_{k+1} + a_k = (\alpha^{k+1} + \beta^{k+1}) + (\alpha^k + \beta^k) = \alpha^k(\alpha+1) + \beta^k(\beta+1) = \alpha^k \cdot \alpha^2 + \beta^k \cdot \beta^2 = \alpha^{k+2} + \beta^{k+2}$.

(VIII) By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer n.

6. **Answer.**

(I) Suppose $\{a_n\}_{n=0}^{\infty}$ is the infinite sequence of real numbers defined by

$$\begin{cases} a_0 &= 1 \\ a_{n+1} &= \dfrac{{a_n}^3}{1+{a_n}^2}\sin^2(a_n) \end{cases}$$

. Denote by $P(n)$ the proposition '$1 \geq a_n > a_{n+1} > 0$'.

(II) Let $k \in \mathbb{N}$. Suppose $P(k)$ is true.

(III) $P(k)$

(IV) $\sin^2(a_{k+1}) > 0$

(V)

$$\begin{aligned} a_{k+1} - a_{k+2} &= a_{k+1} - \frac{{a_{k+1}}^3}{1+{a_{k+1}}^2}\sin^2(a_{k+1}) \\ &= a_{k+1}\left(1 - \frac{{a_{k+1}}^2}{1+{a_{k+1}}^2}\sin^2(a_{k+1})\right) \\ &= a_{k+1}\left(\frac{1}{1+{a_{k+1}}^2} + \frac{{a_{k+1}}^2}{1+{a_{k+1}}^2}\cos^2(a_{k+1})\right) \\ &> 0. \end{aligned}$$

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

7. (a) i. **Answer.**

(I) $\displaystyle\sum_{k=0}^{n}(a_{k+1} - a_k) = a_{n+1} - a_0$

(II) $\displaystyle\sum_{k=0}^{0}(a_{k+1} - a_k) = a_1 - a_0 = a_{0+1} - a_0$.

(III) Suppose $P(m)$ is true

(IV) $a_{m+1} - a_0$

(V) $\displaystyle\sum_{k=0}^{m+1}(a_{k+1} - a_k) = \sum_{k=0}^{m}(a_{k+1} - a_k) + (a_{m+2} - a_{m+1}) = (a_{m+1} - a_0) + (a_{m+2} - a_{m+1}) = a_{m+2} - a_0 =$ $a_{(m+1)+1} - a_0$.

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

ii. ——

(b) **Solution.**

Let $\{c_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers. Let α, β be numbers, with $\alpha \neq 1$. Suppose $c_{n+1} = \alpha c_n + \beta$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, define $a_n = \dfrac{c_n}{\alpha^n}$.

Then by definition, for each $n \in \mathbb{N}$, we have $a_{n+1} = \dfrac{c_n}{\alpha^{n+1}} = \dfrac{\alpha c_n + \beta}{\alpha^{n+1}} = \dfrac{c_n}{\alpha^n} + \dfrac{\beta}{\alpha^{n+1}} = a_n + \dfrac{\beta}{\alpha^{n+1}}$.

By the result described in the statement (A), for each $n \in \mathbb{N}$, we have

$$a_{n+1} - a_0 = \sum_{k=0}^{n}(a_{k+1} - a_k) = \sum_{k=0}^{n} \frac{\beta}{\alpha^{k+1}} = \alpha^{-n-1}\beta \sum_{k=0}^{n} \alpha^k = \alpha^{-n-1}\beta \cdot \frac{1-\alpha^{n+1}}{1-\alpha}$$

Therefore

$$c_{n+1} - \alpha^{n+1}c_0 = \alpha^{n+1}a_{n+1} - \alpha^{n+1}a_0 = \alpha^{n+1}(a_{n+1} - a_0) = \frac{\beta(1-\alpha^{n+1})}{1-\alpha}.$$

Remark. The key step in the application of the telescopic method is displayed below:

By assumption, for each n, we have

$$\left\{\begin{array}{llclcl} & c_{n+1} & - & \alpha & c_n & = \beta \\ \alpha & c_n & - & \alpha^2 & c_{n-1} & = \beta\alpha \\ \alpha^2 & c_{n-1} & - & \alpha^3 & c_{n-2} & = \beta\alpha^2 \\ & \vdots & & & \vdots & \vdots \\ \alpha^{n-2} & c_3 & - & \alpha^{n-1} & c_2 & = \beta\alpha^{n-2} \\ \alpha^{n-1} & c_2 & - & \alpha^n & c_1 & = \beta\alpha^{n-1} \\ \alpha^n & c_1 & - & \alpha^{n+1} & c_0 & = \beta\alpha^n \end{array}\right.$$

Then $c_{n+1} - \alpha^{n+1}c_0 = \sum_{k=0}^{n} \alpha^{n-k}\beta = \sum_{j=0}^{n} \alpha^j \beta = \dfrac{\beta(1-\alpha^{n+1})}{1-\alpha}$.

(c) i. ——

ii. **Solution.**

Let $\theta \in \mathbb{R}$. Suppose $\sin\left(\dfrac{\theta}{2}\right) \neq 0$. Pick any $n \in \mathbb{N}$.

$$\begin{aligned} \left(1 + 2\sum_{k=1}^{n}\cos(k\theta)\right)\sin\left(\frac{\theta}{2}\right) &= \sin\left(\frac{\theta}{2}\right) + \sum_{k=1}^{n} 2\cos(k\theta)\sin\left(\frac{\theta}{2}\right) \\ &= \sin\left(\frac{\theta}{2}\right) + \sum_{k=1}^{n}\left(\sin\left((k+\frac{1}{2})\theta\right) - \sin\left((k-\frac{1}{2})\theta\right)\right) \\ &= \sin\left((n+\frac{1}{2})\theta\right) \end{aligned}$$

By assumption $\sin(\dfrac{\theta}{2}) \neq 0$. Then $= 1 + 2\sum_{k=1}^{n}\cos(k\theta) = \dfrac{\sin((n+1/2)\theta)}{\sin(\theta/2)}$.

iii. ——

iv. ——

8. (a) **Answer.**

i. (I) $|\mu|^2 + |\nu|^2 + 2|\mu| \cdot |\nu| - (\mu+\nu)\overline{(\mu+\nu)} = |\mu|^2 + |\nu|^2 + 2|\mu| \cdot |\overline{\nu}| - \mu\overline{\mu} - \nu\overline{\nu} - \mu\overline{\nu} - \overline{\mu}\nu$

(II) $(\mathsf{Re}(\mu\overline{\nu}))^2$

(III) $(\mathsf{Re}(\mu\overline{\nu}))^2 + (\mathsf{Im}(\mu\overline{\nu}))^2$

(IV) $|\mu+\nu|^2 \leq (|\mu| + |\nu|)^2$

(V) $|\mu| + |\nu| \geq 0$

ii. (I) Suppose $\mu_1, \cdots, \mu_n \in \mathbb{C}$.

(II) $\left|\sum_{j=1}^{n} \mu_j\right| \leq \sum_{j=1}^{n} |\mu_j|$.

(III) $P(2)$ is true

(IV) Let $k \in \mathbb{N} \backslash \{0, 1\}$. Suppose $P(k)$ is true.

(V) $\nu_1, \cdots, \nu_k, \nu_{k+1}$ be complex numbers

(VI)

$$\left|\sum_{j=1}^{k+1} \nu_j\right| = \left|\sum_{j=1}^{k} \nu_j + \nu_{k+1}\right| \leq \left|\sum_{j=1}^{k} \nu_j\right| + |\nu_{k+1}| \leq \sum_{j=1}^{k} |\nu_j| + |\nu_{k+1}| \leq \sum_{j=1}^{k+1} |\nu_j|$$

(VII) the Principle of Mathematical Induction

(b) **Solution.**

i. Let $\zeta \in \mathbb{C}$. Suppose $0 < |\zeta| < 1$. Then we have

$$\left|\sum_{k=1050}^{4060} \zeta^k\right| \leq \sum_{k=1050}^{4060} |\zeta^k| = \sum_{k=1050}^{4060} |\zeta|^k = |\zeta|^{1050} \cdot \sum_{k=0}^{3010} |\zeta|^k = |\zeta|^{1050} \cdot \frac{1 - |\zeta|^{3011}}{1 - |\zeta|} < \frac{|\zeta|^{1050}}{1 - |\zeta|}.$$

The first inequality is a consequence of Statement (T).
The last inequality follows from $|\zeta|^{1050} > 0$ and $0 < |\zeta|^{3011} < 1$.

ii. Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N} \backslash \{0\}$. Suppose $|\alpha| > 5$.

We have $\displaystyle\sum_{k=0}^{n} \frac{5^k}{\alpha^k} = \sum_{k=0}^{n} \left(\frac{5}{\alpha}\right)^k = \frac{1 - (5/\alpha)^{n+1}}{1 - 5/\alpha} = \frac{1 - 5^{n+1}/\alpha^{n+1}}{1 - 5/\alpha}$.

Then $\displaystyle\sum_{k=0}^{n} \frac{5^k}{\alpha^k} - \frac{\alpha}{\alpha - 5} = \sum_{k=0}^{n} \frac{5^k}{\alpha^k} - \frac{1}{1 - 5/\alpha} = -\frac{5^{n+1}/\alpha^{n+1}}{1 - 5/\alpha} = -\frac{5^{n+1}}{\alpha^n(\alpha - 5)}$.

By assumption, $|\alpha| > 5$. Then by Statement (T), we have $|\alpha - 5| \geq |\alpha| - 5 > 0$.

Therefore $\displaystyle\left|\sum_{k=0}^{n} \frac{5^k}{\alpha^k} - \frac{\alpha}{\alpha - 5}\right| = \left|-\frac{5^{n+1}}{\alpha^n(\alpha - 5)}\right| = \frac{5^{n+1}}{|\alpha|^n|\alpha - 5|} \leq \frac{5^{n+1}}{|\alpha|^n(|\alpha| - 5)}$.