1. Let $\zeta = \sin\left(\frac{2\pi}{3}\right) + i\cos\left(\frac{2\pi}{3}\right)$.
   (a) Express ζ in polar form.
   (b) Hence, or otherwise, find the three cubic roots of ζ, expressing your answer in polar form.

2. *In this question, there is no need to justify your answer.*
   (a) Give a full and explicit description of all solutions of the equation $z^5 - 32i = 0$ with complex unknown z. Express your answer in polar form.
   (b) Write down all the complex solutions of the system of inequalities

$$\begin{cases} z^5 - 32i &= 0 \\ \mathsf{Re}(z) &\geq \mathsf{Im}(z) \end{cases}$$

3. Let $f(z)$ be the polynomial given by $f(z) = z^{10} + z^5 + 1$. Let $\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$.
   (a) Write down the quintic roots of ω.
   (b) Express the polynomial $f(z)$ first in the form $(z^5 + P)(z^5 + Q)$, and hence further in the form

$$(z^2 + Az + 1)(z^2 + Bz + 1)(z^2 + Cz + 1)(z^2 + Dz + 1)(z^2 + Ez + 1)$$

   in which P, Q are some appropriate complex numbers, and A, B, C, D, E are some appropriate real numbers. You have to give the values of P, Q, A, B, C, D, E explicitly.
   (c)$^\diamond$ By applying the result above, or otherwise, determine the values of the numbers below. Justify your answers.
   i. $\cos\left(\frac{2\pi}{15}\right) + \cos\left(\frac{8\pi}{15}\right) + \cos\left(\frac{14\pi}{15}\right) + \cos\left(\frac{4\pi}{3}\right) + \cos\left(\frac{26\pi}{15}\right)$.
   ii.

$$\begin{aligned} &\cos\left(\tfrac{2\pi}{15}\right)\cos\left(\tfrac{8\pi}{15}\right) + \cos\left(\tfrac{2\pi}{15}\right)\cos\left(\tfrac{14\pi}{15}\right) + \cos\left(\tfrac{2\pi}{15}\right)\cos\left(\tfrac{4\pi}{3}\right) + \cos\left(\tfrac{2\pi}{15}\right)\cos\left(\tfrac{26\pi}{15}\right) + \cos\left(\tfrac{8\pi}{15}\right)\cos\left(\tfrac{14\pi}{15}\right) \\ + &\cos\left(\tfrac{8\pi}{15}\right)\cos\left(\tfrac{4\pi}{3}\right) + \cos\left(\tfrac{8\pi}{15}\right)\cos\left(\tfrac{26\pi}{15}\right) + \cos\left(\tfrac{14\pi}{15}\right)\cos\left(\tfrac{4\pi}{3}\right) + \cos\left(\tfrac{14\pi}{15}\right)\cos\left(\tfrac{26\pi}{15}\right) + \cos\left(\tfrac{4\pi}{3}\right)\cos\left(\tfrac{26\pi}{15}\right) \end{aligned}$$

4. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (A) and a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)
   (a) Here we prove the statement (A):

   (A) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$ *whenever n is a positive integer.*

> Denote by $P(n)$ the proposition ____(I)____ .
>
> - We have $1 \leq 1 = 2\sqrt{1} - 1$. Hence ____(II)____ .
> - Let k be a positive integer. Suppose ____(III)____ .
>
>   Then $(2\sqrt{k} - 1) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}}\right) \geq$ ____(IV)____ .
>
>   We verify that $P(k+1)$ is true:
>
>   We have
>
> $$(2\sqrt{k+1} - 1) - \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right) = \text{____(V)____} \geq 0$$
>
>   Then $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq$ ____(VI)____ . Hence ____(VII)____ .
>
> ____(VIII)____

   (b) Here we prove the statement (B):

(*B*) *$n(n^2+2)$ is divisible by 3 for any $n \in \mathbb{N}$.*

(I) ______

- We have ______(II)______. By definition of divisibility, ______(III)______. Then $P(0)$ is true.
- ______(IV)______

  Then ___(V)___ is divisible by 3. By definition of divisibility, ______(VI)______.

  We verify that $P(k+1)$ is true:

  We have ______(VII)______. Since q, k are integers, ___(VIII)___ is an integer.

  Then by definition of divisibility, ______(IX)______. Hence $P(k+1)$ is true.

(X) ______

5. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (C) and a proof for the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) We prove the statement (C):

(C) *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of positive real numbers. Suppose $\sum_{j=0}^{n} a_j = \left(\frac{1+a_n}{2}\right)^2$ for each $n \in \mathbb{N}$. Then $a_n = 2n+1$ for each $n \in \mathbb{N}$.*

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of positive real numbers. ______(I)______

Denote by $P(n)$ the proposition ______(II)______.

- We verify that $P(0)$ is true:

  ______(III)______ Hence $P(0)$ is true.
- ______(IV)______

  We verify that $P(k+1)$ is true:

  ______(V)______ Therefore $P(k+1)$ is true.

______(VI)______

(b) We prove the statement (D):

(D) *Let α, β are the two distinct roots of the polynomial $f(x) = x^2 - x - 1$. Suppose $\{a_n\}_{n=1}^{\infty}$ is the infinite sequence of real numbers defined by*

$$\begin{cases} a_1 = 1, & a_2 = 3, \\ & a_{n+2} = a_{n+1} + a_n \quad \text{if} \quad n \geq 1 \end{cases}.$$

Then $a_n = \alpha^n + \beta^n$ for each positive integer n.

______(I)______

Denote by $P(n)$ the proposition '$a_n = \alpha^n + \beta^n$ and $a_{n+1} = \alpha^{n+1} + \beta^{n+1}$'.

- We verify that $P(1)$ is true:

  We have $a_1 =$ ______(II)______. We also have $a_2 =$ ______(III)______.

  Hence $P(1)$ is true.
- ______(IV)______ Then $a_k = \alpha^k + \beta^k$, and $a_{k+1} = \alpha^{k+1} + \beta^{k+1}$.

  We verify that $P(k+1)$ is true:

  We have $a_{k+1} =$ ______(V)______ by ___(VI)___ immediately.

  Now we verify that $a_{(k+1)+1} = \alpha^{(k+1)+1} + \beta^{(k+1)+1}$:

  ______(VII)______

  Therefore $P(k+1)$ is true.

______(VIII)______

6. We introduce (or recall from your *calculus* course) the definition for the notion of *strict monotonicity for infinite sequences of real numbers*:

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.
   - *We say that $\{a_n\}_{n=0}^{\infty}$ is* **strict increasing** *if for any $n \in \mathbb{N}$, $a_n < a_{n+1}$.*
   - *We say that $\{a_n\}_{n=0}^{\infty}$ is* **strictly decreasing** *if for any $n \in \mathbb{N}$, $a_n > a_{n+1}$.*

   Consider the statement (E):

   (E) *Suppose $\{a_n\}_{n=0}^{\infty}$ is the infinite sequence of real numbers defined by*

$$\begin{cases} a_0 & = 1 \\ a_{n+1} & = \dfrac{{a_n}^3}{1+{a_n}^2}\sin^2(a_n) \end{cases}$$

   Then $\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give a proof for the statement (E). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > ________(I)________
   >
   > - We verify that $P(0)$ is true:
   >
   >   We have $a_0 = 1$ and $a_1 = \frac{1}{2}\sin^2(1)$. Then $1 \geq a_0 > a_1 > 0$.
   >
   >   Therefore $P(0)$ is true.
   > - ______(II)______ Then $1 \geq a_k > a_{k+1} > 0$.
   >
   >   We verify that $P(k+1)$ is true:
   >
   >   By __(III)__ , we have $1 \geq a_{k+1} > 0$. Then $0 < \sin^2(a_{k+1}) < 1$ and $0 < \cos^2(a_{k+1}) < 1$.
   >
   >   Since $a_{k+1} > 0$ and $1 + {a_{k+1}}^2 > 0$ and ____(IV)____ , we have $a_{k+2} = \dfrac{{a_{k+1}}^3}{1+{a_{k+1}}^2}\sin^2(a_{k+1}) > 0$.
   >
   >   Also, __________(V)__________ . Then $a_{k+1} > a_{k+2}$.
   >
   >   Now we have $1 \geq a_{k+1} > a_{k+2} > 0$.
   >
   >   Therefore $P(k+1)$ is true.
   >
   > ________(VI)________
   >
   > It follows that $\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.

7. (a) i. Consider the statement (G):

   (G) *Suppose $\{a_n\}_{n=0}^{\infty}$ is an infinite sequence of complex numbers. Then $\sum_{k=0}^{n}(a_{k+1} - a_k) = a_{n+1} - a_0$.*

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give a proof for the statement (G). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > Suppose $\{a_n\}_{n=0}^{\infty}$ is an infinite sequence of complex numbers.
   > Denote by $P(n)$ the proposition ______(I)______ .
   >
   > - We verify that $P(0)$ is true:
   >
   >   We have ________(II)________ . Hence $P(0)$ is true.
   > - Let $m \in \mathbb{N}$. ______(III)______ . Then $\sum_{k=0}^{m}(a_{k+1} - a_k) =$ ______(IV)______ .
   >
   >   We verify $P(m+1)$:
   >
   >   We have ____________(V)____________ . Hence $P(m+1)$ is true.
   >
   > __________(VI)__________

ii. Prove the statement (H):

(H) *Suppose $\{a_n\}_{n=0}^{\infty}$ is an infinite sequence of non-zero complex numbers. Then* $\displaystyle\prod_{k=0}^{n}\frac{a_{k+1}}{a_k}=\frac{a_{n+1}}{a_0}$.

Remarks. The statements (G), (H) give the mechanism for a useful method for computing sums/products of consecutive terms of sequences. This method is known as the **Telescopic Method**.

(b)$^\diamond$ Apply the result described in the statement (G) to prove the statement $(\sharp)$:

$(\sharp)$ *Let $\{c_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers. Let α,β be numbers, with $\alpha\neq 1$. Suppose $c_{n+1}=\alpha c_n+\beta$ for each $n\in\mathbb{N}$. Then $c_n=\alpha^n c_0+\dfrac{\beta(1-\alpha^n)}{1-\alpha}$ for each $n\geq 1$.*

(c) Prove the statements below. (The Telescopic Method may be useful.)

i. *Let $\theta\in\mathbb{R}$. Suppose $\sin(\theta)\neq 0$. Then $\cos(\theta)\cos(2\theta)\cos(2^2\theta)\cdot\ldots\cdot\cos(2^n\theta)=\dfrac{\sin(2^{n+1}\theta)}{2^{n+1}\sin(\theta)}$ for any $n\in\mathbb{N}$.*

ii. *Let $\theta\in\mathbb{R}$. Suppose $\sin\left(\dfrac{\theta}{2}\right)\neq 0$. Then $1+2\displaystyle\sum_{k=1}^{n}\cos(k\theta)=\dfrac{\sin((n+1/2)\theta)}{\sin(\theta/2)}$ for any $n\in\mathbb{N}$.*

iii. *Let $\theta\in\mathbb{R}$. Suppose $\sin(2^p\theta)\neq 0$ for any $p\in\mathbb{N}$. Then $\displaystyle\sum_{k=0}^{n}2^k\tan(2^k\theta)=\cot(\theta)-2^{n+1}\cot(2^{n+1}\theta)$ for any $n\in\mathbb{N}$.*

iv. *Let $\theta\in\mathbb{R}$. Suppose $\sin(2^p\theta)\neq 0$ for any $p\in\mathbb{N}$. Then $\displaystyle\sum_{k=1}^{n}\csc(2^k\theta)=\cot(\theta)-\cot(2^n\theta)$ for any $n\in\mathbb{N}\backslash\{0\}$.*

8. (a) Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (K), and a proof for the statement (L). Both statements are referred to as (the 'non-strict inequality part' of) the **Triangle Inequality 'on the complex plane'**. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. We prove the statement (K):

(K) *Suppose $\mu,\nu\in\mathbb{C}$. Then $|\mu+\nu|\leq|\mu|+|\nu|$.*

> Suppose $\mu,\nu\in\mathbb{C}$. We have
>
> $(|\mu|+|\nu|)^2-|\mu+\nu|^2 =$ ____(I)____ $=2|\mu\overline{\nu}|-2\mathsf{Re}(\mu\overline{\nu})=2(|\mu\overline{\nu}|-\mathsf{Re}(\mu\overline{\nu}))$ ——$(\star)$
>
> Note that ____(II)____ $\leq$ ____(III)____ $=|\mu\overline{\nu}|^2$.
> Then $\mathsf{Re}(\mu\overline{\nu})\leq|\mathsf{Re}(\mu\overline{\nu})|\leq|\mu\overline{\nu}|$. Therefore $|\mu\overline{\nu}|-\mathsf{Re}(\mu\overline{\nu})\geq 0$.
> Then by $(\star)$, ____(IV)____. Since $|\mu+\nu|\geq 0$ and ____(V)____, we have $|\mu+\nu|\leq|\mu|+|\nu|$.

ii. We prove the statement (L):

(L) *Let $n\in\mathbb{N}\backslash\{0,1\}$. Suppose $\mu_1,\mu_2,\cdots,\mu_n\in\mathbb{C}$. Then $\left|\displaystyle\sum_{j=1}^{n}\mu_j\right|\leq\displaystyle\sum_{j=1}^{n}|\mu_j|$.*

> Denote by $P(n)$ the proposition below:
>
> ____(I)____. Then ____(II)____.
>
> - By the statement (K), ____(III)____.
> - ____(IV)____
>
>   We verify that $P(k+1)$ is true:
>
>   Suppose ____(V)____. Then ____(VI)____
>
>   Therefore $P(k+1)$ is true.
>
> By ____(VII)____, $P(n)$ is true for any $n\in\mathbb{N}\backslash\{0,1\}$.

(b) By applying the results above, or otherwise, prove the statements below:

i. *Let $\zeta\in\mathbb{C}$. Suppose $0<|\zeta|<1$. Then $\left|\displaystyle\sum_{k=1050}^{4060}\zeta^k\right|<\dfrac{|\zeta|^{1050}}{1-|\zeta|}$.*

ii.$^\diamond$ *Let $\alpha\in\mathbb{C}$, and $n\in\mathbb{N}\backslash\{0\}$. Suppose $|\alpha|>5$. Then $\left|\displaystyle\sum_{k=0}^{n}\frac{5^k}{\alpha^k}-\frac{\alpha}{\alpha-5}\right|\leq\dfrac{5^{n+1}}{|\alpha|^n(|\alpha|-5)}$.*