1. (a) Fill in the blanks in the passage below so as to give the definition for the notion of *rational numbers*:

   Suppose $x \in \mathbb{R}$. Then we say that x is **rational** *if* ____(I)____ *such that* ____(II)____ .

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A) and a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   i. Here we prove the statement (A):

   (A) *Let $x, y \in \mathbb{R}$. Suppose x, y are rational. Then $x + y$ is rational.*

   > Let $x, y \in \mathbb{R}$. ____(I)____ .
   >
   > [We want to deduce that $x+y$ is rational. This amounts to verifying the statement '*there exist some $s, t \in \mathbb{Z}$ such that $t \neq 0$ and $s = t(x + y)$*'.]
   >
   > By definition, ____(II)____ $m, n \in \mathbb{Z}$ such that ____(III)____ $m = nx$.
   >
   > Also, ____(IV)____ $q \neq 0$ and $p = qy$.
   >
   > Note that $mq + pn = nxq + qyn = nq(x + y)$.
   >
   > Since ____(V)____ and ____(VI)____ , we have $nq \neq 0$.
   >
   > Also, since $m, n, p, q \in \mathbb{Z}$, we have ____(VII)____ .
   >
   > Hence, by definition, ____(VIII)____ .

   ii. Here we prove the statement (B):

   (B) *Let x, y be real numbers. Suppose x, y are rational. Then xy is rational.*

   > Let x, y be real numbers. Suppose x, y are rational.
   >
   > [We want to deduce that xy is rational. This amounts to verifying the statement '*there exist some $s, t \in \mathbb{Z}$ such that $t \neq 0$ and $s = t(xy)$*'.]
   >
   > By definition, ____(I)____ $n \neq 0$ and $m = nx$.
   >
   > Also, ____(II)____ .
   >
   > Note that ____(III)____ .
   >
   > Since $n \neq 0$ ____(IV)____ , we have ____(V)____ .
   >
   > Also, ____(VI)____ , we have $mp \in \mathbb{Z}$ and $nq \in \mathbb{Z}$.
   >
   > Hence, by definition, xy is rational.

2. (a) Explain the phrase *divisibility for integers* by stating the appropriate definition.

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (C) and a proof for the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   i. Here we prove the statement (C):

   (C) *Let $x, y, n \in \mathbb{Z}$. Suppose x is divisible by n and y is divisible by n. Then $x + y$ is divisible by n.*

   > ____(I)____
   >
   > Since x is divisible by n, ____(II)____ .
   >
   > Since y is divisible by n, ____(III)____ .
   >
   > We have ____(IV)____ . Then $x + y = kn + \ell n = (k + \ell)n$.
   >
   > Since $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, we have ____(V)____ .
   >
   > Therefore, by definition, ____(VI)____ .

   ii. Here we prove the statement (D):

   (D) *Let $x, y, n \in \mathbb{Z}$. Suppose x is divisible by n or y is divisible by n. Then xy is divisible by n.*

(I) ______

- (Case 1). Suppose ______ (II) ______. Then ______ (III) ______ $x = kn$.

  Note that ______ (IV) ______. Also, ______ (V) ______.

  Then ______ (VI) ______.

- (Case 2). ______ (VII) ______ y is divisible by n. Modifying the argument for (Case 1), we also deduce that ______ (VIII) ______.

Hence, ______ (IX) ______.

3. Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (E), a proof for the statement (F), and a proof for the statement (G). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) Here we prove the statement (E):

   (E) *Let a, b be real numbers. Suppose $a > b > 0$. Then $\sqrt{a^2 - b^2} + \sqrt{2ab - b^2} > a$.*

   Let a, b be real numbers. Suppose $a > b > 0$.
   Further suppose that ______ (I) ______.

   [*Reminder.* Under what we have supposed and what we have further supposed, we try to obtain a contradiction.]

   Note that $\sqrt{a^2 - b^2} \geq 0$ ______ (II) ______. Then $a \geq \sqrt{a^2 - b^2} + \sqrt{2ab - b^2}$ ______ (III) ______ 0.
   Since $a > b > 0$, we have $a^2 - b^2 = (a - b)(a + b) \geq 0$. Then $(\sqrt{a^2 - b^2})^2 =$ ______ (IV) ______.
   Similarly, ______ (V) ______. Then $(\sqrt{2ab - b^2})^2 = 2ab - b^2$.
   Therefore we would have

   $$\begin{aligned} \underline{\quad (VI) \quad} &\geq \underline{\quad (VII) \quad} \\ &= (a^2 - b^2) + (2ab - b^2) + 2\sqrt{(a^2 - b^2)(2ab - b^2)} \\ &= a^2 - 2b^2 + 2ab + 2\sqrt{(a - b)(a + b)(2a - b)b}. \end{aligned}$$

   Hence

   $$0 \leq \underline{\quad (VIII) \quad} \leq \underline{\quad (IX) \quad} = b(b - a).$$

   Recall that by assumption, $a > b > 0$. Then ______ (X) ______.
   Therefore $0 \leq b(b - a) < 0$. Contradiction arises.
   It follows that, in the first place, $\sqrt{a^2 - b^2} + \sqrt{2ab - b^2} > a$.

   (b) Here we prove the statement (F):

   (F) *Let $m, n \in \mathbb{Z}$. Suppose $0 < |m| < |n|$. Then m is not divisible by n.*

   Let ______ (I) ______. Suppose ______ (II) ______.
   Further suppose ______ (III) ______.

   [*Reminder.* Under what we have supposed and what we have further supposed, we try to obtain a contradiction.]

   Since m was divisible by n, by definition ______ (IV) ______.
   By assumption, ______ (V) ______. Then $m \neq 0$.
   Since $m \neq 0$ and $m = kn$, we would have ______ (VI) ______. Then $|k| \neq 0$.
   ______ (VII) ______, $|k|$ would also be an integer. Then $|k| \geq 1$.
   By assumption $|n| > 0$. Then $|m| = |kn| =$ ______ (VIII) ______ $= |n|$.
   Also, by ______ (IX) ______, $|n| > |m|$.
   Then $|m| \geq |n| > |m|$. Therefore $|m| > |m|$. Contradiction arises.
   It follows that, in the first place, ______ (X) ______.

(c) Here we prove the statement (G):

(G) *Let x be a positive real number. Suppose x is irrational. Then $\sqrt{x}$ is irrational.*

> Let x be a positive real number.
> Suppose ______(I)______ .
> Further suppose ______(II)______ .
>
> [*Reminder.* Under what we have supposed and what we have further supposed, we try to obtain a contradiction.]
>
> Since ______(III)______ , we have $(\sqrt{x})^2 =$ ___(IV)___ .
> Since ______(V)______ , $(\sqrt{x})^2$ would be rational as well.
> Therefore x would be ______(VI)______ .
> By assumption, x is ______(VII)______ . Then x would be simultaneously ______(VIII)______ ______(IX)______ .
> It follows that, in the first place, ______(X)______ .

4. (a) Explain the phrase *common divisor for integers* by stating the appropriate definition.
   (b) Explain the phrase *prime number* by stating the appropriate definition.
   (c) State, without proof, Euclid's Lemma.
   (d) Here we prove the statement (H), with the help of Euclid's Lemma:

   (H) *$\sqrt[3]{3}$ is irrational.*

> ______(I)______ .
> Then $\sqrt[3]{3}$ would be a rational number. Therefore ______(II)______ such that ______(III)______ .
> Without loss of generality, we may assume that m, n have no common divisors other than $1, -1$.
> Since $m = n \cdot \sqrt[3]{3}$, we would have $m^3 = 3n^3$.
> Note that n^3 was an integer. Then ______(IV)______ .
> Now also note that 3 is a prime number. Then, by ______(V)______ , m would be divisible by 3.
> Therefore ______(VI)______ .
> Then we would have $27k^3 = (3k)^3 = m^3 = 3n^3$. Therefore $n^3 = 9k^3 = 3(3k^3)$.
> ______(VII)______
> Note that ______(VIII)______ . Then, by Euclid's Lemma, ______(IX)______ .
> Therefore both m, n would be divisible by 3. Hence 3 would be a common divisor of m, n.
> Recall that we have assumed that ______(X)______ . Contradiction arises.
>
> Therefore the assumption that $\sqrt[3]{3}$ was not irrational is false. It follows that $\sqrt[3]{3}$ is irrational in the first place.

5.[◇] We introduce (or recall from your *calculus* course) the definitions on *boundedness* for infinite sequences of real numbers:

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

- *$\{a_n\}_{n=0}^{\infty}$ is said to be* **bounded above in** $\mathbb{R}$ *if the statement* (BoAb) *holds:*

  (BoAb) *There exists some $\kappa \in \mathbb{R}$ such that for any $n \in \mathbb{N}$, $a_n \leq \kappa$.*

  Such a real number κ, if it exists, is called an **upper bound** *of $\{a_n\}_{n=0}^{\infty}$.*
- *$\{a_n\}_{n=0}^{\infty}$ is said to be* **bounded below in** $\mathbb{R}$ *if the statement* (BoBe) *holds:*

  (BoBe) *There exists some $\lambda \in \mathbb{R}$ such that for any $n \in \mathbb{N}$, $a_n \geq \lambda$.*

  Such a real number λ, if it exists, is called an **lower bound** *of $\{a_n\}_{n=0}^{\infty}$.*
- *$\{a_n\}_{n=0}^{\infty}$ is said to be* **bounded in** $\mathbb{R}$ *if $\{a_n\}_{n=0}^{\infty}$ is both bounded above in $\mathbb{R}$ and bounded below in $\mathbb{R}$.*

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (I), and a proof for the statement (J).

(a) Here we prove the statement (I):

(I) *Let* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *be infinite sequences of real numbers. Suppose* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *are bounded above in* $\mathbb{R}$. *Then* $\{a_n + b_n\}_{n=0}^{\infty}$ *is bounded above in* $\mathbb{R}$.

> Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be infinite sequences of real numbers.
> ______(I)______
>
> [We want to deduce that $\{a_n + b_n\}_{n=0}^{\infty}$ *is bounded above in* $\mathbb{R}$. This amounts to verifying that *there exists some* $\mu \in \mathbb{R}$ *such that for any* $n \in \mathbb{N}$, $a_n + b_n \leq \mu$.]
>
> Since $\{a_n\}_{n=0}^{\infty}$ is ___(II)___, there exists some $\kappa \in \mathbb{R}$ such that ___(III)___ —— $(*)$
> Since $\{b_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$, ___(IV)___ such that ___(V)___, $b_n \leq \lambda$. —— $(*')$
> Define $\mu = \kappa + \lambda$. By definition, since $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we have $\mu \in \mathbb{R}$.
>
> [For such a number μ, we verify that for any $n \in \mathbb{N}$, $a_n + b_n \leq \mu$.]
>
> Pick any ___(VI)___.
> For this n, by $(*)$, we have ___(VII)___. —— $(**)$
> For the same n, by $(**)$, we also have ___(VIII)___. —— $(**')$
> Then by $(**)$, $(**')$, we have ___(IX)___ for the same n.
> Therefore, by definition, ___(X)___.

(b) Here we prove the statement (J):

(J) *Suppose* $\{a_n\}_{n=0}^{\infty}$ *is an infinite sequence of real numbers. Then* $\{a_n\}_{n=0}^{\infty}$ *is bounded in* $\mathbb{R}$ *iff there exists some* $\nu \in \mathbb{R}$ *such that for any* $n \in \mathbb{N}$, $|a_n| \leq \nu$.

> Suppose $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.
>
> - Suppose ___(I)___.
>
>   [We deduce that *there exists some* $\nu \in \mathbb{R}$ *such that for any* $n \in \mathbb{N}$, $|a_n| \leq \nu$.]
>
>   By definition, $\{a_n\}_{n=0}^{\infty}$ is both ___(II)___.
>
>   ___(III)___ $\{a_n\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$, ___(IV)___.
>
>   Since ___(V)___, there exists some $\lambda \in \mathbb{R}$ such that ___(VI)___.
>
>   Define $\nu = |\kappa| + |\lambda|$. By definition, since $\kappa, \lambda \in \mathbb{R}$, we have $\nu \in \mathbb{R}$.
>
>   [For such a number ν, we verify that for any $n \in \mathbb{N}$, $|a_n| \leq \nu$.]
>
>   Pick any $n \in \mathbb{N}$.
>
>   For this n, we have $-(|\kappa| + |\lambda|) \leq$ ___(VII)___ $\leq \kappa \leq |\kappa| \leq$ ___(VIII)___.
>
>   Then ___(IX)___.
>
> - ___(X)___ there exists some $\nu \in \mathbb{R}$ such that for any $n \in \mathbb{N}$, $|a_n| \leq \nu$.
>
>   [We deduce that $\{a_n\}_{n=0}^{\infty}$ *is bounded in* $\mathbb{R}$.]
>
>   We verify that $\{a_n\}_{n=0}^{\infty}$ is ___(XI)___:
>
>   * Define $\kappa = |\nu|$. By definition, since $\nu \in \mathbb{R}$, we have $\kappa \in \mathbb{R}$.
>   ___(XII)___
>   Therefore, by definition, $\{a_n\}$ is bounded above in $\mathbb{R}$.
>
>   We verify that $\{a_n\}_{n=0}^{\infty}$ is bounded below in $\mathbb{R}$:
>
>   * ___(XIII)___. By definition, since $\nu \in \mathbb{R}$, we have $\lambda \in \mathbb{R}$.
>   ___(XIV)___
>   Therefore, by definition, ___(XV)___.