1. *This is a review question on* solving equations/inequalities *which can be handled with purely algebraic manipulations.*

   Solve each of the equations/inequalities/systems below for all its real solutions. 'Check solution' when indeed you have to do so. [1]

   (a) $x + \sqrt{x+1} = 11$.

   (b) $2(4^x + 4^{-x}) - 7(2^x + 2^{-x}) + 10 = 0$.

   (c) $\log_{5-x}(215 - x^3) = 3$.

   (d) $|x^2 - 5x + 6| = x$.

   (e) $x|x| + 5x + 6 = 0$.

   (f) $(x-4)^2 - 5|x-4| + 6 = 0$.

   (g) $\left\{\begin{array}{rcrcl} xy & + & x & = & 6 \\ xy & - & y & = & 2 \end{array}\right.$

   (h) $\left\{\begin{array}{lcl} xy & = & 35 \\ x^{\log_5(y)} & = & 7 \end{array}\right.$

   (i) $x^2 - 3x < 10$.

   (j) $\left\{\begin{array}{lcl} (x+1)(x-6) & \geq & 8 \\ 3x - 1 & \geq & 5 \end{array}\right.$

   (k) $(x+1)^2 > 16$ or $2x + 5 > 7$.

   (l) $(x-1)(x-2)(x-3) \geq 0$.

   (m) $\dfrac{2}{3-x} \leq 1$.

   (n) $2x - \dfrac{3}{x} \geq 1$.

   (o) $\dfrac{x^2-1}{x^2-4} \leq -2$.

   (p) $|x^2 - 5x| < 6$.

   (q) $\left|\dfrac{3x+11}{x+2}\right| < 2$.

   (r)$^\diamondsuit$ $\big|\ |x| - 4\ \big| > 3$.

   (s) $|x^2 - 3| \leq 2|x|$.

   (t)$^\diamondsuit$ $|2x+1| < 3x - 2$.

   Remark. Now suppose you are not required to give any step of algebraic manipulation. Can you modify the 'graphical method' for solving equations in *school mathematics* to determine the answer for each part as quickly as possible?

2. *This is a review question on* quadratic polynomials.

   Let a, b, c, r be numbers, with $a \neq 0$ and $c \neq 0$ and $r \neq 0$. Let $f(x)$ be the quadratic polynomial given by $f(x) = ax^2 + bx + c$. Suppose α, β are the roots of $f(x)$. Further suppose $\alpha = r\beta$.

   Prove that $rb^2 = (Pr + Q)^2 ac$. Here P, Q are some integers whose values you have to determined explicitly.

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[1] In various situations, you may need apply some special rules about the words '*and*', '*or*', known as the *Distributive Laws for 'and', 'or'*, (with or without your being aware of them). They may be in-formally stated as below:

A. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*

* *(blah-blah-blah or bleh-bleh-bleh) and bloh-bloh-bloh.*
* *(blah-blah-blah and bloh-bloh-bloh) or (bleh-bleh-bleh and bloh-bloh-bloh).*

B. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*

* *(blah-blah-blah and bleh-bleh-bleh) or bloh-bloh-bloh.*
* *(blah-blah-blah or bloh-bloh-bloh) and (bleh-bleh-bleh or bloh-bloh-bloh).*

More will be said of them in the discussion on *logic*.

3. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (A), a proof for the statement (B), a proof for the statement (C) and a proof for the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Here we prove the statement (A):

(A) *Let $x, y \in \mathbb{R}$. Suppose $x + y > 1$ and $x > y$. Then $x^2 - y^2 > x - y$.*

> Let $x, y \in \mathbb{R}$. ______(I)______ .
> We note that $(x - y)(x + y) - (x - y) =$ ______(II)______ . —— $(\star)$
> ___(III)___ $x + y > 1$, we have $x + y - 1 > 0$.
> Since ______(IV)______ , we have $x - y > 0$.
> Since ______(V)______ , we have $(x - y)(x + y - 1) > 0$.
> Then, by $(\star)$, ______(VI)______ .
> Therefore ______(VII)______ .

(b) Here we prove the statement (B):

(B) *Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$. Then $x^3 + y^3 \geq xy(x + y)$.*

> ______(I)______ .
> We have $(x^3 + y^3) - xy(x + y) =$ ______(II)______ . —— $(\star)$
> Since $x > 0$ and $y > 0$, we have $x + y > 0$.
> Since x, y are real numbers, ______(III)______ . Then $(x - y)^2 \geq 0$.
> Since $x + y > 0$ and $(x - y)^2 \geq 0$, we have $(x + y)(x - y)^2$ ___(IV)___ .
> Then, by $(\star)$, ______(V)______ .
> Hence $x^3 + y^3 \geq xy(x + y)$.

(c) Here we prove the statement (C):

(C) *Let $x, y, z \in \mathbb{R}$. Suppose $x^2 + y^2 + z^2 + xy - yz - xz \leq 0$. Then $x = y = z = 0$.*

> Let $x, y, z \in \mathbb{R}$. ______(I)______
> By assumption, we have $2(x^2 + y^2 + z^2 + xy - yz - xz) \leq$ ___(II)___ . —— $(\star)$
> Note that $2(x^2 + y^2 + z^2 + xy - yz - xz) = \left(x^2 + y^2 + 2xy\right) +$ ______(III)______ $=$ ______(IV)______ . —— $(\star\star)$
> Since x, y, z are real numbers, each of $x + y, y - z, x - z$ is a real number.
> Then $(x + y)^2 \geq 0$ and ______(V)______ respectively. —— $(\star\star\star)$
> Now by $(\star)$, $(\star\star)$, $(\star\star\star)$, we have $0 \geq$ ______(VI)______ $\geq (x + y)^2 \geq$ ___(VII)___ .
> Then $(x + y)^2 = 0$.
> Modifying the argument above, we also deduce ______(VIII)______ .
> Therefore ______(IX)______ .
> Since $y - z = 0$ and $x - z = 0$, we have $x = y = z$.
> Since $x + y = 0$ and $x = y$, we have ___(X)___ . Then, since $x = z$, we have ___(XI)___ also.

(d) Here we prove the statement (D):

(D) *Let $x, y, z \in \mathbb{R}$. Suppose $y > x > 0$ and $z > -y$. Then $\dfrac{x + z}{y + z} > \dfrac{x}{y}$ iff $z > 0$.*

Let $x, y, z \in \mathbb{R}$. ______ (I) ______ .

Since __ (II) __ , we have $y + z > 0$. Then, since $y > 0$ also, we have $y(y + z)$ __ (III) __ .

- [We want to deduce: '*If* $z > 0$ *then* $\frac{x + z}{y + z} > \frac{x}{y}$.']

  __ (IV) __ $z > 0$.

  Then, since $z > 0$ and $y > x$, we have ______ (V) ______ .

  Therefore $(x + z)y = xy + zy > xy + zx = x(y + z)$.

  Then $\frac{x + z}{y + z} - \frac{x}{y} =$ ______ (VI) ______ .

  Therefore $\frac{x + z}{y + z} > \frac{x}{y}$.

- [We want to deduce: '*If* $\frac{x + z}{y + z} > \frac{x}{y}$ *then* $z > 0$.']

  ______ (VII) ______ .

  Then $xy + zy = (x + z)y =$ ______ (VIII) ______ $= x(y + z) = xy + zx$.

  Therefore $z(y - x) =$ ______ (IX) ______ .

  Then ($z > 0$ __ (X) __ $y - x > 0$) or (______ (XI) ______).

  Since $y > x$, we have $y - x > 0$.

  Hence $z > 0$ and $y - x > 0$. In particular, $z > 0$.

It follows that ______ (XII) ______ .

4.◇ We introduce/recall the definitions on *strict monotonicity* for real-valued functions of one real variable:

> *Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.*
>
> - *h is said to be* **strictly increasing** *on I if the statement* (StrIncr) *holds:*
>
>   (StrIncr) *For any $s, t \in I$, if $s < t$ then $h(s) < h(t)$.*
>
> - *h is said to be* **strictly decreasing** *on I if the statement* (StrDecr) *holds:*
>
>   (StrDecr) *For any $s, t \in I$, if $s < t$ then $h(s) > h(t)$.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (E) and a proof for the statement (F). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^4$ for any $x \in \mathbb{R}$.

Here we prove the statement (E):

(E) *f is strictly increasing on the interval* $[0, +\infty)$.

[We are going to verify the statement (†): '*For any* $s, t \in [0, +\infty)$, *if* $s < t$ *then* $f(s) < f(t)$.']

Pick any $s, t \in [0, +\infty)$. ___(I)___ $s < t$.

[We want to deduce $f(t) - f(s) > 0$.]

We have $f(t) - f(s) =$ ___(II)___. — ($\star$)

[We want to check that each of $t - s$, $t + s$, $t^2 + s^2$ is positive.
First we ask whether it is true that $t - s > 0$.]

Since ___(III)___, we have $t - s > 0$.

[Next we ask whether it is true that $t + s > 0$.]

Since ___(IV)___ $s < t$, we have $t > 0$.
Then, since $s \geq 0$ and $t > 0$, we have ___(V)___.

[Finally we ask whether it is true that $t^2 + s^2 > 0$.]

Since $t > 0$, we have ___(VI)___. Since ___(VII)___, we have $s^2 \geq 0$. Then ___(VIII)___.
Now, since $t - s > 0$ and $t + s > 0$ and $t^2 + s^2 > 0$, we have $(t - s)(t + s)(t^2 + s^2) > 0$.
Then by ($\star$), we have ___(IX)___.
Therefore $f(s) < f(t)$.
It follows from definition that ___(X)___.

(b) Here we prove the statement (F):

(F) *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function. Define the function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $g(x) = f(x) - 2x^3$ *for any* $x \in \mathbb{R}$. *Suppose* f *is strictly decreasing on* $\mathbb{R}$. *Then* g *is strictly decreasing on* $\mathbb{R}$.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function.
Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by $g(x) = f(x) - 2x^3$ for any $x \in \mathbb{R}$.
___(I)___

[We are going to verify (possibly with the help of the assumption 'f *is strictly decreasing on* $\mathbb{R}$') the statement (†): '*For any* $s, t \in \mathbb{R}$, *if* $s < t$ *then* $g(s) > g(t)$.']

___(II)___

[We want to deduce $g(s) - g(t) > 0$.]

We have ___(III)___. — ($\star$)
___(IV)___, we have $f(s) - f(t) > 0$.

Since $s < t$, we have $t - s > 0$. Also, ___(V)___ $= \dfrac{1}{2}[t^2 + (t + s)^2 + s^2] \geq$ ___(VI)___.
Then $2(t - s)(t^2 + st + s^2) \geq 0$.
Since $f(s) - f(t) > 0$ and $2(t - s)(t^2 + st + s^2) \geq 0$, we have ___(VII)___.
Then by ($\star$), we have $g(s) - g(t) > 0$.
Therefore ___(VIII)___.
It follows from definition that g is strictly decreasing on $\mathbb{R}$.