1. **Definition of surjectivity**.

   Let A, B be sets, and $f : A \longrightarrow B$ be a function from A to B.

   f is said to be **surjective** if the statement (S) holds:

   (S): For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.

   In the symbols of logic, (S) is:

   $$(\forall y \in B)[(\exists x \in A)(y = f(x))].$$

   So for each $y \in B$, (S) gives an existence statement.

2. **Pictorial visualizations of surjective functions**.

   (a) 'Blobs-and-arrows diagram'.

      f is surjective exactly when:

      • every element of B is 'pointed' at by at least one element of A.

   (b) 'Coordinate plane diagram'.

      f is surjective exactly when:

      • for each $b \in B$, the equation $b = f(u)$ with 'unknown' u has at least one solution in A.
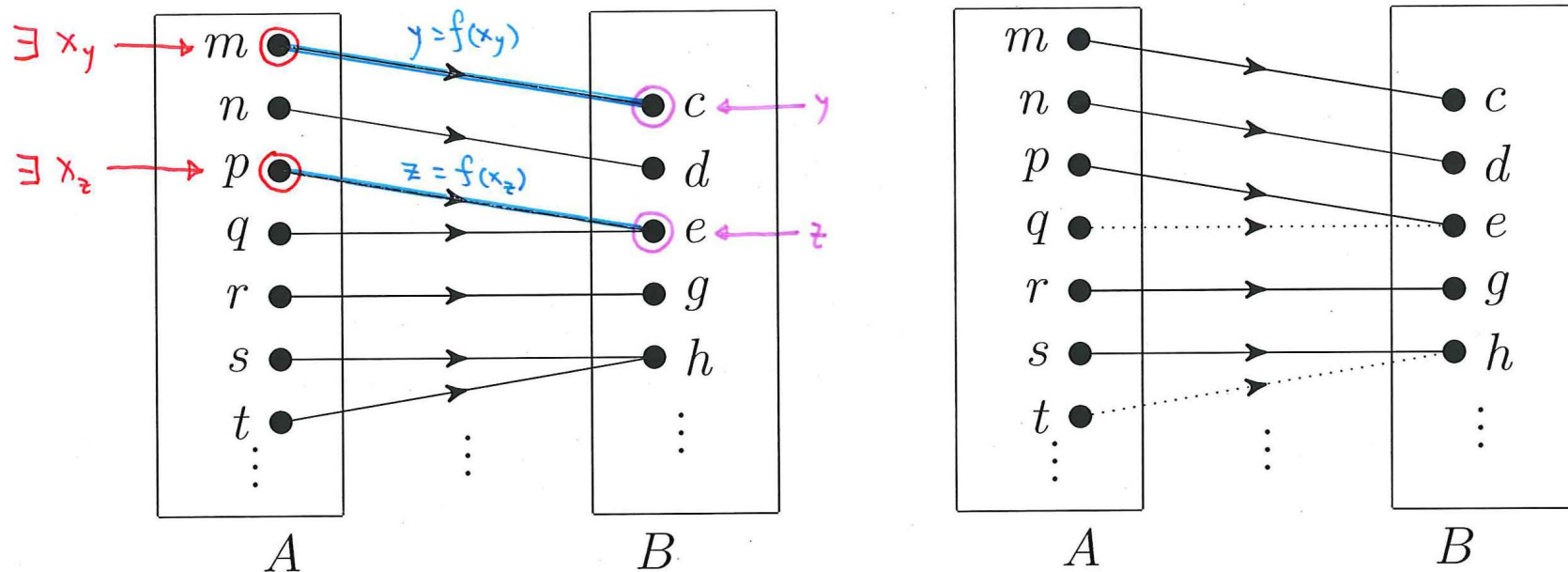
3. Example of a surjective function and its pictorial visualizations.

Let $A = \{m, n, p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h \cdots\}$, and $f : A \longrightarrow B$ be the surjective function defined by

$$f(m) = c, \ f(n) = d, \ f(p) = f(q) = e, \ f(r) = g, \ f(s) = f(t) = h, \cdots .$$
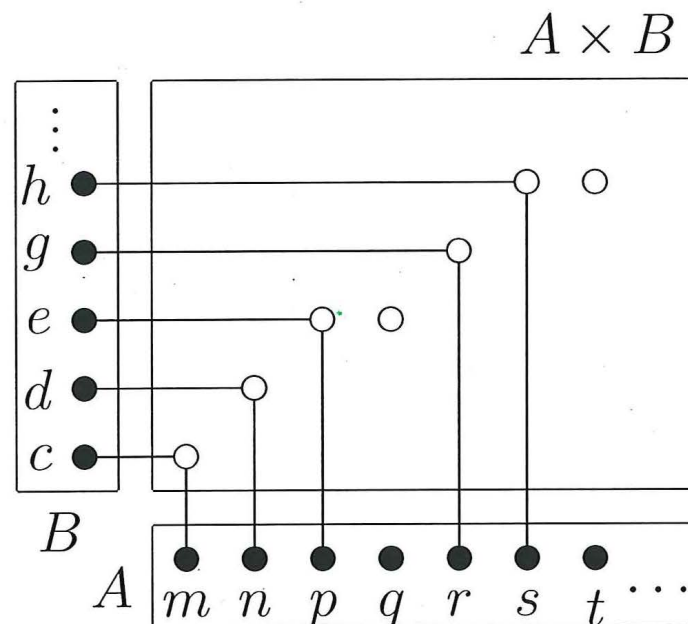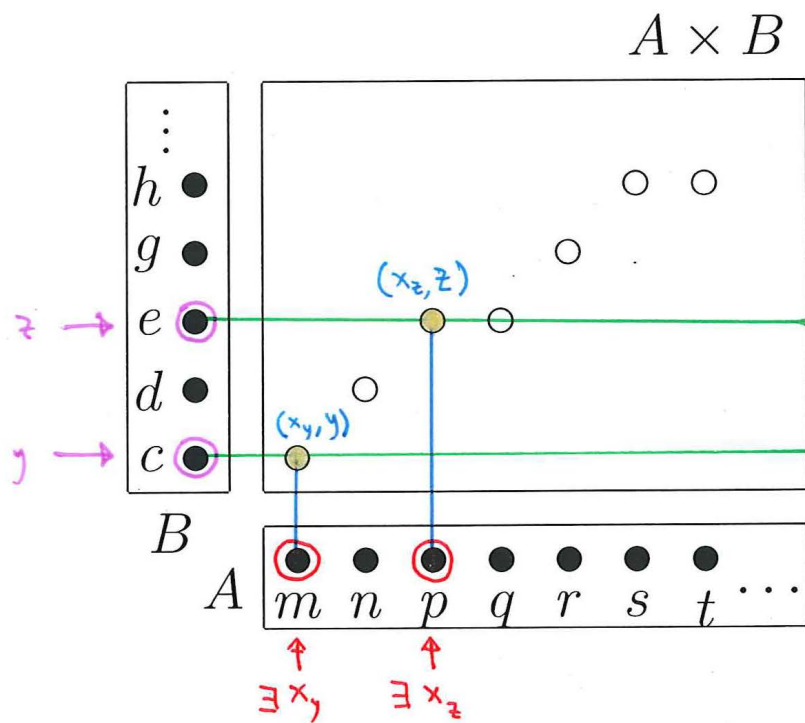
(a) 'Blobs-and-arrows diagram'.

Every element of B is 'pointed' at by at least one element of A.

(b) *'Coordinate plane diagram'.*

For each $b \in B$, the equation $b = f(u)$ with 'unknown' u has at least one solution in A.

Horizontal line "Constant c"
cuts the graph of f somewhere,
namely, (m, c).

$\Longleftrightarrow$ The equation $c = f(u)$ with unknown u in A
has some solution, namely, '$u = m$'.

There is the same phenomenon for every element of B, not just c.

4. **How to check that a given function $f : A \longrightarrow B$ is surjective?**

What to check?

(S): *For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.*

Procedure:

(1) Pick any $y \in B$. (From this moment on this y is fixed.)

(2a) Name an appropriate x_y which you believe will satisfy both $x_y \in A$ and $y = f(x_y)$, when it is immediately clear which x_y is to be named.

(2b) When it is not immediately clear which x_y is to be named, do some roughwork:

* With y fixed, solve the equation $y = f(u)$ with 'unknown' u in A. One such solution is an appropriate candidate x_y to be named.

(3) Verify that the x_y just named indeed satisfy $x_y \in A$ and $y = f(x_y)$.

5. **Negation of surjectivity**.

Let A, B be sets, and $f : A \longrightarrow B$ be a function. $f : A \longrightarrow B$ is not surjective iff the statement $\sim(S)$ holds:

$\sim(S)$: There exists some $y_0 \in B$ such that for any $x \in A$, $y_0 \neq f(x)$.

In the symbols of logic, $\sim(S)$ is:

$$(\exists y_0 \in B)[(\forall x \in A)(y_0 \neq f(x))].$$

6. **Pictorial visualizations of non-surjective functions**.

(a) 'Blobs-and-arrows diagram'.

f is not surjective exactly when:

- some element of B is 'pointed' at by no element of A.

(b) 'Coordinate plane diagram'.

f is not surjective exactly when:

- there is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has no solution in A.
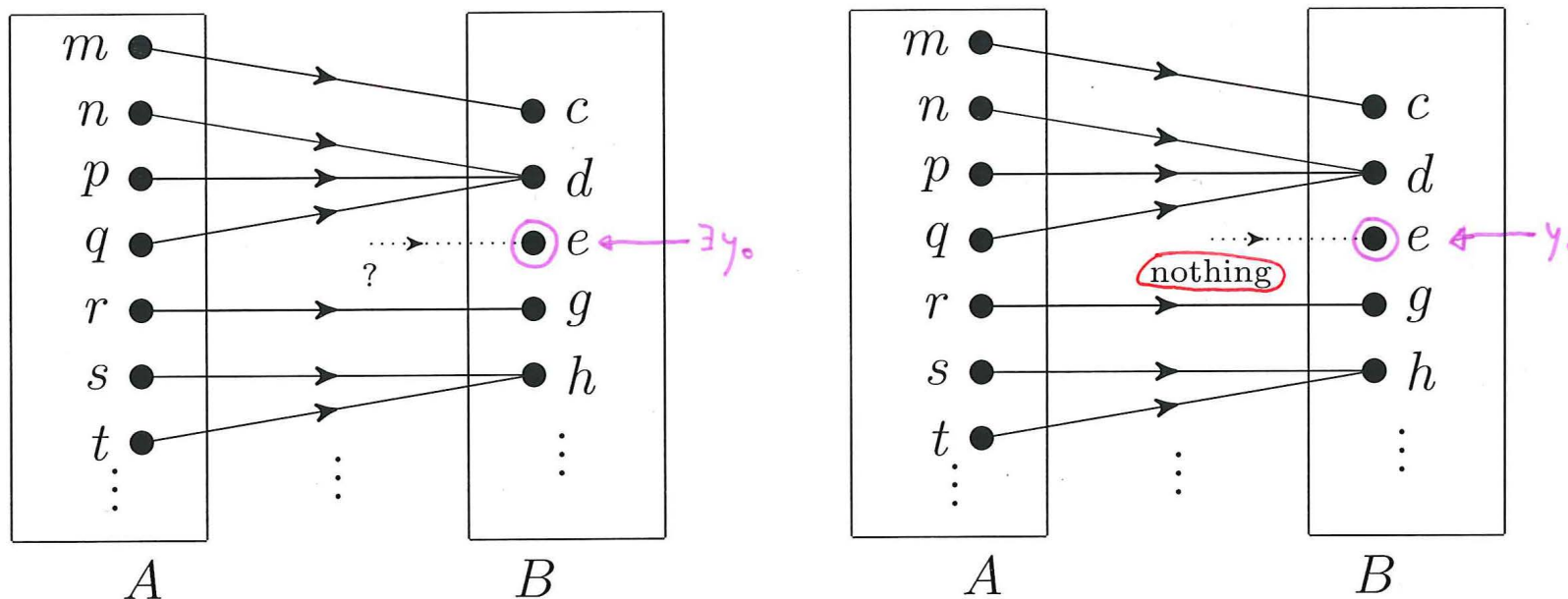
7. Example of a non-surjective functions and its pictorial visualizations.

Let $A = \{m, n, p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h \cdots\}$, and $f : A \longrightarrow B$ be the non-surjective function defined by

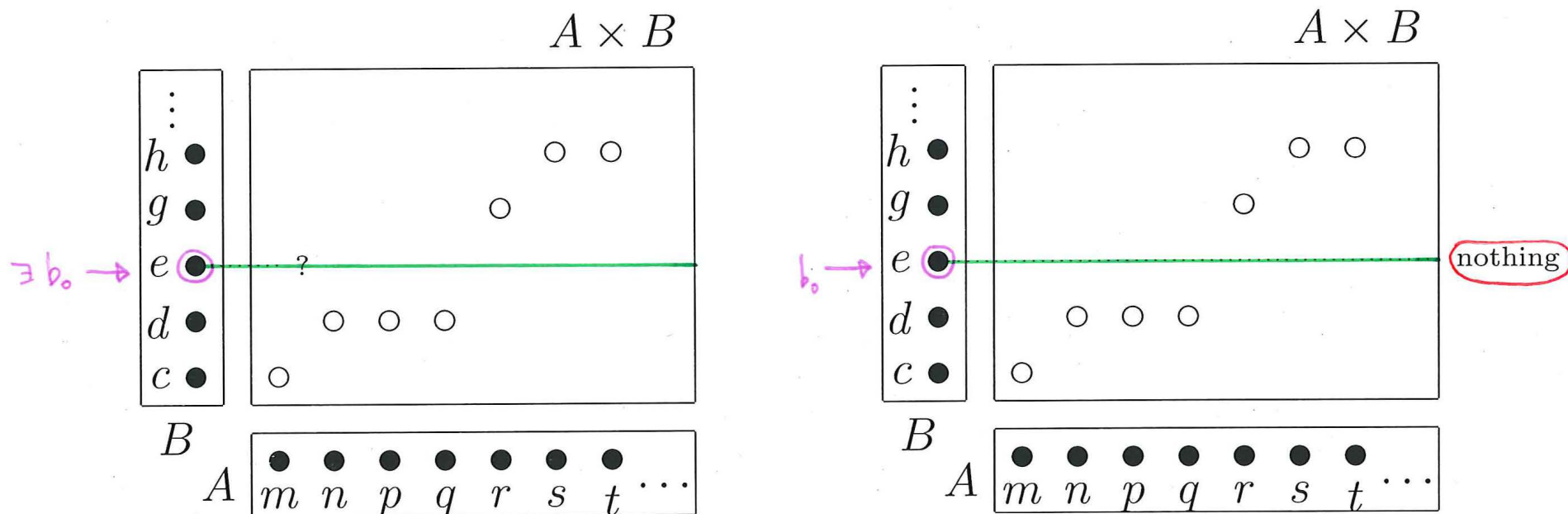$$f(m) = c, f(n) = f(p) = f(q) = d, f(r) = g, f(s) = f(t) = h, \cdots .$$

(a) 'Blobs-and-arrows diagram'.

Some element of B is 'pointed' at by no element of A.

(b) 'Coordinate plane diagram'.

There is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has no solution in A.

'Horizontal line "Constant e"'
cuts the graph of f nowhere.

$\Longleftrightarrow$ The equation $e = f(u)$ with unknown u in A has no solution.

8. **How to check that a given function $f : A \longrightarrow B$ is not surjective?**

What to check?

$\sim(S)$: *There exists some $y_0 \in B$ such that for any $x \in A$, $y_0 \neq f(x)$.*

Procedure:

(1a) Name appropriate 'concrete' $y_0 \in B$ which you believe will satisfy $y_0 \neq f(x)$ for any $x \in A$, when it is immediately clear which y_0 to be named.

(1b) When it is not immediately clear which y_0 is to be named, do some roughwork:

(1b.i) First look for a necessary condition in terms of the 'value' of b for the equation $b = f(u)$ with 'unknown' u to have a solution, by asking this question:

'What can be said of b if $b = f(u)$ has a solution?'

(1b.ii) Look for an appropriate y_0 which has to be something failing to satisfy this necessary condition.

(2) Then go to (†) or (‡):

(†) Verify that for any $x \in A$, $f(x) \neq y_0$.

(‡) Obtain a contradiction under the assumption 'there existed some $x_0 \in A$ such that $f(x_0) = y_0$'.

9. **Definition of injectivity**.

Let A, B be sets, and $f : A \longrightarrow B$ be a function from A to B.

f is said to be **injective** if the statement (I) holds:

(I): For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.

Various re-formulation of (I):

(I'): For any $x, w \in A$, if $x \neq w$ then $f(x) \neq f(w)$.

(I''): For any $y \in B$, for any $x, w \in A$, if $(y = f(x)$ and $y = f(w))$ then $x = w$.

In the symbols of logic, (I), (I'), (I'') are:

$$(I) : \quad (\forall x \in A)(\forall w \in A)[(f(x) = f(w)) \rightarrow (w = x)].$$
$$(I') : \quad (\forall x \in A)(\forall w \in A)[(x \neq w) \rightarrow (f(x) \neq f(w))].$$
$$(I'') : \quad (\forall y \in B)(\forall x \in A)(\forall w \in A)[[(y = f(x)) \wedge (y = f(w))] \rightarrow (x = w)].$$

So for each $y \in B$, (I'') gives a uniqueness statement.

10. **Negation of injectivity**.

Let A, B be sets, and $f : A \longrightarrow B$ be a function.

f is not injective iff the statement $\sim(I)$ holds:

$\sim(I)$: There exist some $x_0, w_0 \in A$ such that $f(x_0) = f(w_0)$ and $x_0 \neq w_0$.

Re-formulation of $\sim(I)$ as $\sim(I'')$:

$\sim(I'')$: There exist some $y_0 \in B$, $x_0, w_0 \in A$ such that $(y_0 = f(x_0)$ and $y_0 = f(w_0)$ and $x_0 \neq w_0)$.

$\sim(I)$, $\sim(I')$ are exactly the same as each other. (Why?)

In the symbols of logic, $\sim(I)$, $\sim(I')$, $\sim(I'')$ are:

$\sim(I), \sim(I') :$ $\quad (\exists x_0 \in A)(\exists w_0 \in A)[(f(x_0) = f(w_0)) \wedge (x_0 \neq w_0)].$

$\sim(I'') :$ $\quad (\exists y_0 \in B)(\exists x_0 \in A)(\exists w_0 \in A)[(y_0 = f(x_0)) \wedge (y_0 = f(w_0)) \wedge (x_0 \neq w_0)].$

11. **Pictorial visualizations of non-injective functions**.

   (a) 'Blobs-and-arrows diagram'.

   f is not injective exactly when:

   - some element of B is 'pointed' at by two or more elements of A.

   (b) 'Coordinate plane diagram'.

   f is not injective exactly when:

   - there is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has two or more solutions in A.
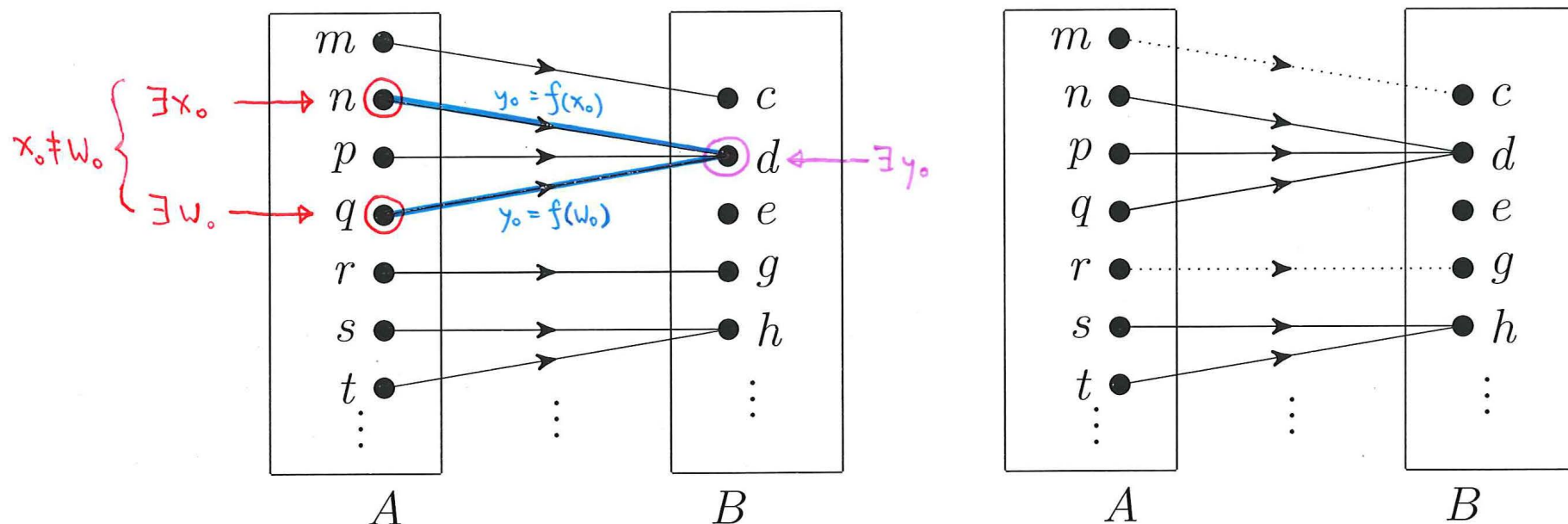
12. **Example of a non-injective function and its pictorial visualizations**.

Let $A = \{m, n, p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h \cdots\}$, and $f : A \longrightarrow B$ be the non-injective function defined by

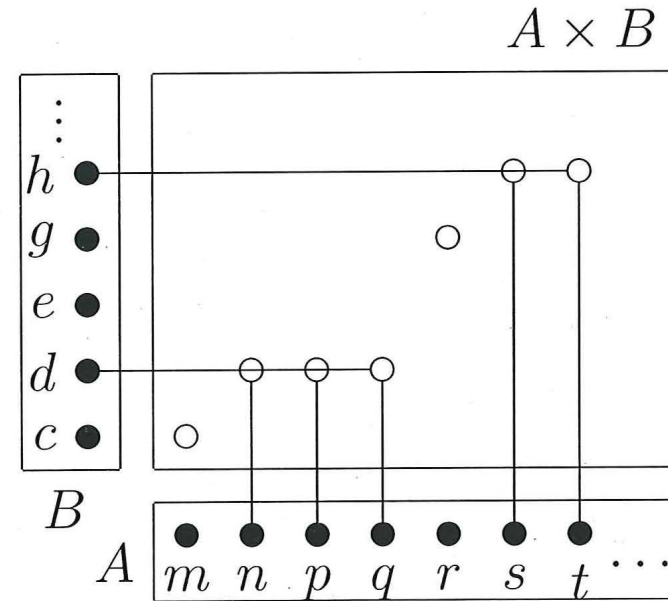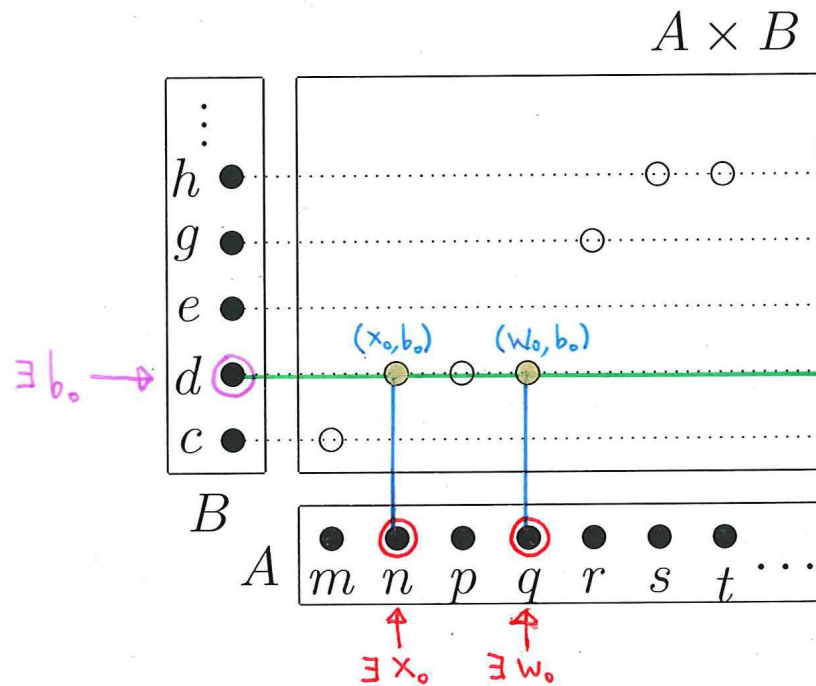$$f(m) = c, f(n) = f(p) = f(q) = d, f(r) = g, f(s) = f(t) = h, \cdots .$$

(a) 'Blobs-and-arrows diagram'.

Some element of B is 'pointed' at by two or more elements of A.

(b) *'Coordinate plane diagram'.*

There is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has two or more solutions in A.

$A \times B$ $A \times B$

$\exists b_0 \longrightarrow$

(x_0, b_0) (w_0, b_0)

B

$A \quad m \quad n \quad p \quad q \quad r \quad s \quad t \; \cdots$

$\exists x_0 \qquad \exists w_0$

'Horizontal line "Constant d"'
cuts the graph of f at two distinct points,
namely $(n, d), (q, d)$.

$\Longleftrightarrow$ The equation $d = f(u)$ with unknown u in A
has two distinct solutions, namely '$u = n$', '$u = q$'.

13. **Pictorial visualizations of injective functions**.

    (a) 'Blobs-and-arrows diagram'.

        f is injective exactly when:

         ● every element of B is 'pointed' at by at most one element of A.

    (b) 'Coordinate plane diagram'.

        f is injective exactly when:

         ● for each $b \in B$, the equation $b = f(u)$ with 'unknown' u has at most one solution in A.
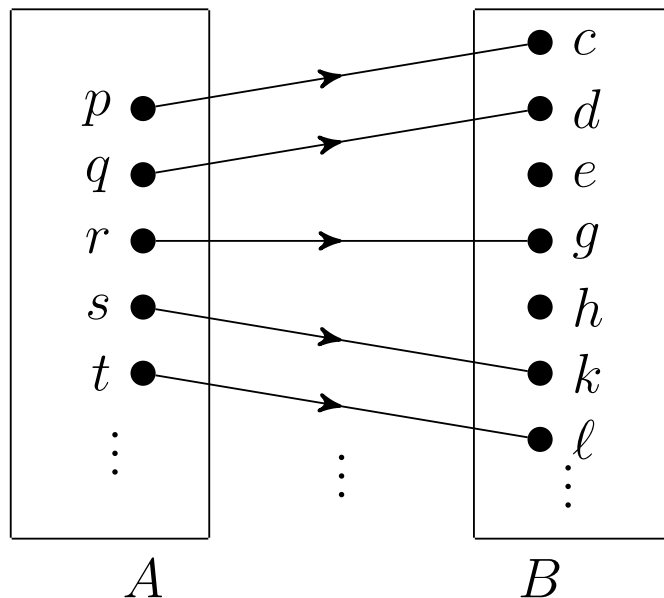
14. **Example of an injective function and its pictorial visualization**.

Let $A = \{p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h, k, \ell \cdots\}$, and $f : A \longrightarrow B$ be the injective function defined by

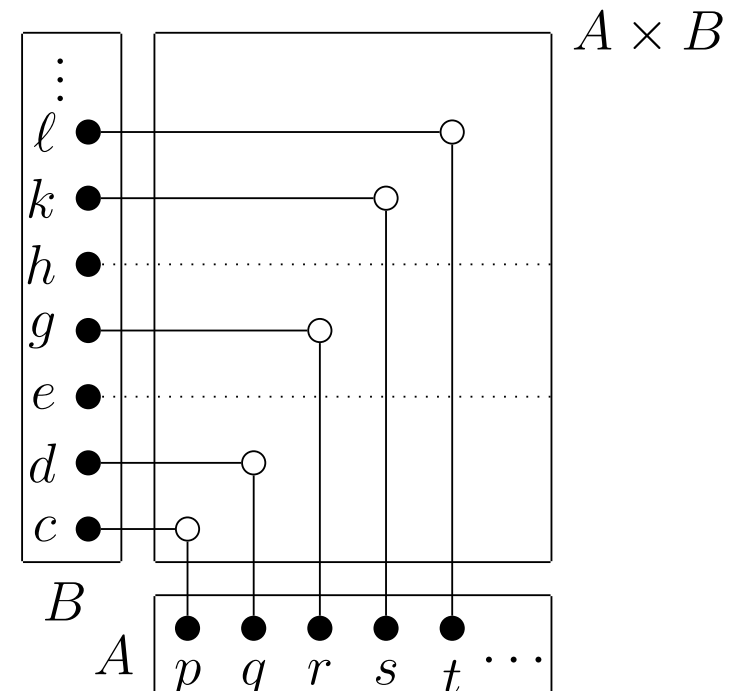$$f(p) = c,\ f(q) = d,\ f(r) = g,\ f(s) = k,\ f(t) = \ell, \cdots.$$

(a) *Blobs-and-arrows diagram*.

Every element of B is 'pointed' at by at most one element of A.

(b) *'Coordinate plane diagram'*.

For each $b \in B$, the equation $b = f(u)$ with 'unknown' u has at most one solution in A.

15. **How to check that a given function $f : A \longrightarrow B$ is injective?**

   What to check?

   (I): *For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.*

   Procedure:

   (1) Pick any $x, w \in A$. (From this moment on, x, w are fixed.)

   (2) Suppose $f(x) = f(w)$. Then verify that $x = w$.

16. **How to check that a given function $f : A \longrightarrow B$ is not injective?**

   What to check?

   $\sim(I)$: *There exist some $x_0, w_0 \in A$ such that $f(x_0) = f(w_0)$ and $x_0 \neq w_0$.*

   Procedure:

   (1a) Name appropriate 'concrete' distinct $x_0, w_0 \in A$ which you believe will satisfy $f(x_0) = f(w_0)$, when it is immediately clear which x_0, w_0 are to be named.

   (1b) When it is not immediately clear which distinct x_0, w_0 are to be named, do some roughwork:

   * Look for an appropriate $b_0 \in B$ for which the equation $b_0 = f(u)$ with 'unknown' u has at least two solutions in A. Two such solutions are to be named as x_0, w_0.

   (2) Verify that $f(x_0) = f(w_0)$ indeed.