1. **Statements with many quantifiers**.

When we carefully analyse the logical structure of a mathematical statement, say, S, we will most likely find that S is of the form

$$(\mathfrak{q}_x x)((\mathfrak{q}_y y)(\cdots((\mathfrak{q}_z z)((\mathfrak{q}_w w)P(x, y, \cdots, z, w)))\cdots)),$$

in which:

- $P(x, y, \cdots, z, w)$ is a predicate with variables $x, y, \cdots, z, w$, and
- each of $\mathfrak{q}_x$, $\mathfrak{q}_y$, ..., $\mathfrak{q}_z$, $\mathfrak{q}_w$ stands for the universal quantifier $\forall$ or the existential quantifier $\exists$.

How to obtain S from $P(x, y, \cdots, z, w)$? 'Close the variables $w, z, \cdots, y, x$ with quantifiers' one by one:

- $P(x, y, \cdots, z, w)$,
- $(\mathfrak{q}_w w)P(x, y, \cdots, z, w)$,
- $(\mathfrak{q}_z z)((\mathfrak{q}_w w)P(x, y, \cdots, z, w))$,
- ...
- $(\mathfrak{q}_y y)(\cdots((\mathfrak{q}_z z)((\mathfrak{q}_w w)P(x, y, \cdots, z, w)))\cdots)$,
- $(\mathfrak{q}_x x)((\mathfrak{q}_y y)(\cdots((\mathfrak{q}_z z)((\mathfrak{q}_w w)P(x, y, \cdots, z, w)))\cdots))$.

2. **Statements starting with two quantifiers.**

From a predicate $Q(x, y)$ with two variables x, y, eight statements can be formed:

(1) $(\forall x)[(\forall y)Q(x, y)]$.

(2) $(\forall y)[(\forall x)Q(x, y)]$.

(3) $(\exists x)[(\exists y)Q(x, y)]$.

(4) $(\exists y)[(\exists x)Q(x, y)]$.

(5) $(\forall x)[(\exists y)Q(x, y)]$.

(6) $(\exists y)[(\forall x)Q(x, y)]$.

(7) $(\exists x)[(\forall y)Q(x, y)]$.

(8) $(\forall y)[(\exists x)Q(x, y)]$.

We accept (1), (2) to be logically equivalent. Examples:

(a) *For any $x > 0$, for any $y > 0$, $x + y > 0$.* $(\forall x)\Big[(\forall y)\Big(\big[(x>0)\wedge(y>0)\big] \rightarrow (x+y>0)\Big)\Big]$.

(b) *Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $|x| = |y|$.*

'$y|x$'

$(\forall x)\Big[(\forall y)\Big(\big[(x\in\mathbb{Z})\wedge(y\in\mathbb{Z})\wedge(y|x)\wedge(x|y)\big] \rightarrow (|x|=|y|)\Big)\Big]$.

We accept (3), (4) to be logically equivalent (in most situations). Examples:

(a) *There exist some irrational numbers x, y such that $x + y$ is a rational number.*

(b) *There exist some integers q, r such that $10000 = 333q + r$ and $0 \leq r \leq 332$.*

$(\exists q)\Big[(\exists r)\Big((q\in\mathbb{Z})\wedge(r\in\mathbb{Z})\wedge(10000=333q+r)\wedge(0\leq r\leq 332)\Big)\Big]$.

$(\exists x)\Big[(\exists y)\Big((x\in\mathbb{R}\setminus\mathbb{Q})\wedge(y\in\mathbb{R}\setminus\mathbb{Q})\wedge(x+y\in\mathbb{Q})\Big)\Big]$.

Care must be taken with (5), (6), (7), (8).

3. **Statements starting with one universal quantifier and one existential quantifier.**

Non-mathematical examples.

Compare and contrast the statements in each pair ($\flat$), ($\sharp$) below:

(a)($\flat$) *Every student gets A in some MATH course.*
(No big deal; everyone has his/her own 'lucky' course.)

($\sharp$) *In some MATH course, every student gets A.*
(Then you will rush to enrole in such a course.)

Waiting list : 999/999
CUSIS status : closed

(b)($\flat$) *In every MATH course, some student gets A.*
(No big deal; you don't expect us to be excessively harsh.)

($\sharp$) *Some student gets A in every MATH course.*
(Then you will look for 'source' from him/her.)

(!)

$Q(x, y)$: Student x gets A in MATH course y.

(a) ($\flat$) : $(\forall x)((\exists y)\, Q(x, y))$

($\sharp$) : $(\exists y)((\forall x)\, Q(x, y))$

(b) ($\flat$) : $(\forall y)((\exists x)\, Q(x, y))$

($\sharp$) : $(\exists x)((\forall y)\, Q(x, y))$

Something seemingly special happens.

$Q(x, y)$: Student x gets F in MATH course y.

Now replace 'A' by 'F', and compare and contrast the resultant statements.

(a')(♭) *Every student gets F in some MATH course.* $(\forall x)((\exists y)\, Q(x, y))$
(Then getting F is nothing, but you still hope it happens to you no more than once.)

(♯) *In some MATH course, every student gets F.* $(\exists y)((\forall x)\, Q(x, y))$
(!) → (Then you will hope this is not a compulsory course.)

(b')(♭) *In every MATH course, some student gets F.* $(\forall y)((\exists x)\, Q(x, y))$
(Then you will work hard and pray you are not those hopefully very few ones.)

(♯) *Some student gets F in every MATH course.* $(\exists x)((\forall y)\, Q(x, y))$
(!) → (You will probably not find him/her as a classmate next year.)

'Moral of the story': Be careful with the 'relative positioning' of the universal and existential quantifiers.

Mathematical Examples.

Compare and contrast the statements in each pair ($\flat$), ($\sharp$)below:

$Q(x,y) : x < y.$

(c)($\flat$) *For any $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that $x < y$.*

($\sharp$) *There exists some $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, $x < y$.*

$(\flat) : (\forall x \in \mathbb{R})\big((\exists y \in \mathbb{R})\, Q(x,y)\big).$

$(\sharp) : (\exists y \in \mathbb{R})\big((\forall x \in \mathbb{R})\, Q(x,y)\big).$

($\flat$) is true. Justification:

- Pick any $x \in \mathbb{R}$. Take $y = x+1$. Then $y \in \mathbb{R}$ and $x < x+1 = y$.

($\sharp$) is false. Justification:

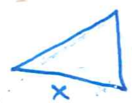
- Suppose there existed some $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, $x < y$. Then, since $y \in \mathbb{R}$, we would have $y < y$. Contradiction arises.

(d)($\flat$) *For any triangle x, there exists some circle y such that y passes through all three vertices of x.*

($\sharp$) *There exists some circle y such that for any triangle x, y passes through all three vertices of x.*

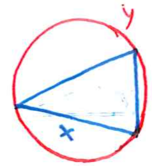
($\flat$) is true. [Check what you learnt about circumcircles for triangles:

Given:



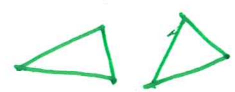
...

Then:



What y exactly is depends on what x is.]

($\sharp$) is false. [Is there any circle that could simultaneously pass through all six vertices of these two triangles?


]

$Q(x,y): x+y=x$.

(e)($\flat$) *For any $x \in \mathbb{Z}$, there exists some $y \in \mathbb{Z}$ such that $x+y=x$.* ($\flat$) : $(\forall x \in \mathbb{Z})((\exists y \in \mathbb{Z}) Q(x,y))$.

($\sharp$) *There exists some $y \in \mathbb{Z}$ such that for any $x \in \mathbb{Z}$, $x+y=x$.* ($\sharp$) : $(\exists y \in \mathbb{Z})((\forall x \in \mathbb{Z}) Q(x,y))$.

($\flat$) is true. Justification:

- Pick any $x \in \mathbb{Z}$. Take $y=0$. Then $y \in \mathbb{Z}$ and $x+y=x+0=x$.

But ($\flat$) is not useful.

($\sharp$) is true and useful: it pinpoints the special nature of the integer 0:

Regardless of the value of $x \in \mathbb{Z}$, we have $x+0=x$.

(f)($\flat$) *(Let S be a non-empty subset of $\mathbb{N}$.) For any $x \in S$, there exists some $y \in S$ such that $y \leq x$.*

($\sharp$) *(Let S be a non-empty subset of $\mathbb{N}$.) There exists some $y \in S$ such that for any $x \in S$, $y \leq x$.*

($\flat$) is true. Justification:

- Pick any $x \in S$. Take $y=x$. Then $y \in S$ and $y=x \leq x$.

But ($\flat$) is not useful.

($\sharp$) is (believed to be) true.

It is the Well-ordering Principle for Integers.

In each of these examples, we have a pair of statements of the form:

($\flat$) $(\forall x)[(\exists y)Q(x,y)]$. ($\sharp$) $(\exists y)[(\forall x)Q(x,y)]$.

They are resultants of different 'sequences' in 'closing variables with quantifiers':

- How to obtain ($\flat$)? First $Q(x,y)$; next $(\exists y)Q(x,y)$; finally $(\forall x)[(\exists y)Q(x,y)]$.
- How to obtain ($\sharp$)? First $Q(x,y)$; next $(\forall x)Q(x,y)$; finally $(\exists y)[(\forall x)Q(x,y)]$.

The convention for ($\flat$) to be understood is:

- For any object x, there exists some object y_x, *depending* on what x is (as indicated by the subscript 'x' in 'y_x') such that $Q(x, y_x)$ is a true statement.

The convention for ($\sharp$) to be understood is:

- There exists some object y such that for any object x, $Q(x,y)$ is a true statement.

If (for some very good reason,) you need start with 'for any ojbect x' in a 'wordy' formulation of ($\sharp$), you must write in this way:

- For any object x, there exists some object y *independent* of the choice of x such that $Q(x,y)$ is a true statement.

Warning. Always remember these points when you read or write a statement involving both the universal quantifier and the existential quantifier:

(a) The statements ($\flat$), ($\sharp$) are different.

- ($\sharp$) implies ($\flat$): if ($\sharp$) is true then ($\flat$) is true.
- However, ($\flat$) does not imply ($\sharp$): when ($\flat$) is true, ($\sharp$) may be true or false.

(b) The 'relative positioning' of '$\forall x$', '$\exists y$' cannot be interchanged.

- In ($\flat$), y 'depends' on x.
- In ($\sharp$), y does not 'depend' on x.

(c) If you are in doubt, recall some examples which help you distinguish the meanings of ($\flat$) and ($\sharp$). For instance, refer to 'non-mathematical examples'.

(d) Ask yourself whether what you write is the same as what you will be understood. For instance, if what you mean is

'*for any* $x \in \mathbb{R}$, *there exists some* $y \in \mathbb{R}$ *such that* $x < y$', ← $(\forall x \in \mathbb{R})((\exists y \in \mathbb{R})(x < y))$

do not write the statement as

'*for any* $x \in \mathbb{R}$, $x < y$ *for some* $y \in \mathbb{R}$', ← Ambiguity: Is it '(for any $x \in \mathbb{R}$, $x < y$) for some $y \in \mathbb{R}$' or 'for any $x \in \mathbb{R}$, ($x < y$ for some $y \in \mathbb{R}$)'?

or worse,

'*there exists some* $y \in \mathbb{R}$ *such that* $x < y$ *for any* $x \in \mathbb{R}$'.

This is to be understood as: $(\exists y \in \mathbb{R})((\forall x \in \mathbb{R})(x < y))$

4. **Negations of statements starting with two quantifiers**.

We apply the rules for negating statements with one quantifier repeatedly for statements with two quantifiers:

(a) The negation of '$(\forall x)[(\exists y)Q(x,y)]$' is

$$\text{'}(\exists x)[(\forall y)(\sim Q(x,y))]\text{'.}$$

(b) The negation of '$(\exists y)[(\forall x)Q(x,y)]$'

$$\text{is '}(\forall y)[(\exists x)(\sim Q(x,y))]\text{'.}$$

(c) The negation of '$(\forall x)[(\forall y)Q(x,y)]$' is

$$\text{'}(\exists x)[(\exists y)(\sim Q(x,y))]\text{'.}$$

(d) The negation of '$(\exists x)[(\exists y)Q(x,y)]$' is

$$\text{'}(\forall x)[(\forall y)(\sim Q(x,y))]\text{'.}$$

Examples. How to write down the negations of the statements below?

(a) *There exists some $y \in S$ such that for any $x \in T$, $x < y$.* —— $(*)$

Convert the statement to be negated into a 'chain of symbols':

$(*)$ reads : $(\exists y \in S)((\forall x \in T)\ (x < y))$.

Now repeatedly apply the rules for negating statements with one quantifier:

Negation of $(*)$?
- $\sim[(\exists y \in S)((\forall x \in T)\ (x < y))]$
- $(\forall y \in S)[\sim((\forall x \in T)\ (x < y))]$
- $(\forall y \in S)((\exists x \in T)[\sim(x < y)])$

'$\sim(x<y)$' is the same as '$x \geq y$'

Now convert this last 'chain of symbols' into words:

For any $y \in S$, there exits some $x \in T$ such that $x \geq y$.

(b) *For any $a, b \in \mathbb{Z}$, $a + b$ is divisible by 2.* —— $(*)$

$(*)$ reads : $(\forall a \in \mathbb{Z})((\forall b \in \mathbb{Z})$ ($a+b$ is divisible by 2)).

Negation of $(*)$?
- $\sim[(\forall a \in \mathbb{Z})((\forall b \in \mathbb{Z})$ ($a+b$ is divisible by 2))].
- $(\exists a \in \mathbb{Z})[\sim((\forall b \in \mathbb{Z})$($a+b$ is divisible by 2))].
- $(\exists a \in \mathbb{Z})((\exists b \in \mathbb{Z})[\sim$($a+b$ is divisible by 2)]).

In words, the negation of $(*)$ reads: there exist some $a, b \in \mathbb{Z}$ such that $a+b$ is not divisible by 2.

(c) *For any* $z \in \mathbb{C}$, *there exists some* $w \in \mathbb{R}$ *such that* $\mathrm{Re}(z+w) = \mathrm{Im}(z+w)$. —— (*)

(*) reads: $(\forall z \in \mathbb{C})\left((\exists w \in \mathbb{R})\left(Re(z+w) = Im(z+w)\right)\right)$

Negation of (*)? • $\sim\left[(\forall z \in \mathbb{C})\left((\exists w \in \mathbb{R})\left(Re(z+w) = Im(z+w)\right)\right)\right]$

• $(\exists z \in \mathbb{C})\left[\sim\left((\exists w \in \mathbb{R})\left(Re(z+w) = Im(z+w)\right)\right)\right]$

• $(\exists z \in \mathbb{C})\left((\forall w \in \mathbb{R})\underbrace{\left[\sim\left(Re(z+w) = Im(z+w)\right)\right]}\right)$

'$\sim(Re(z+w) = Im(z+w))$' is the same as '$Re(z+w) \neq Im(z+w)$'.

In words, the negation of (*) reads: There exists some $z \in \mathbb{C}$ such that for any $w \in \mathbb{R}$, $Re(z+w) \neq Im(z+w)$.

(d) *There exist some* $s, t \in \mathbb{Q}$ *such that* ($s + t \in \mathbb{Z}$ and $st \notin \mathbb{Z}$). —— (*)

(*) reads: $(\exists s \in \mathbb{Q})((\exists t \in \mathbb{Q})(s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z}))$.

Negation of (*)? • $\sim\left[(\exists s \in \mathbb{Q})\left((\exists t \in \mathbb{Q})(s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z})\right)\right]$.

• $(\forall s \in \mathbb{Q})\left[\sim\left((\exists t \in \mathbb{Q})(s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z})\right)\right]$.

• $(\forall s \in \mathbb{Q})\left((\forall t \in \mathbb{Q})\underbrace{\left[\sim(s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z})\right]}\right)$.

'$\sim(s+t \in \mathbb{Z} \text{ and } st \notin \mathbb{Z})$' is the same as '$s+t \notin \mathbb{Z}$ or $st \in \mathbb{Z}$'.

In words, the negation of (*) reads: For any $s, t \in \mathbb{Q}$, ($s+t \notin \mathbb{Z}$ or $st \in \mathbb{Z}$).

5. **Statements with many quantifiers**.

The principles in the discussion above can be extended to statements with three or more quantifiers.

Questions. How to read and/or write them? How to negate them?

6. **Examples from linear algebra.**

(a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and S be a subset of $\mathbb{R}^n$.

How to formulate $\underbrace{\text{'}\textit{every vector in } S \textit{ is a linear combination of } \mathbf{u}, \mathbf{v}, \mathbf{w} \textit{ over } \mathbb{R}\text{'}}_{(\star)}$?

- Formulation in words:

$(\star)$ reads: For any $x \in S$, there exist some $a, b, c \in \mathbb{R}$ such that $x = au + bv + cw$.

- Formulation in symbols:

$(\star)$ reads: $(\forall x \in S)[(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\exists c \in \mathbb{R})\ (x = au + bv + cw)]$

How to formulate $\underbrace{\text{'}\textit{not every vector in } S \textit{ is a linear combination of } \mathbf{u}, \mathbf{v}, \mathbf{w} \textit{ over } \mathbb{R}\text{'}}_{\text{Negation of } (\star)}$?

- Formulation in symbols:

Negation of $(\star)$ reads: $(\exists x \in S)[(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})\ (x \neq au + bv + cw)]$

- Formulation in words:

Negation of $(\star)$ reads: There exists some $x \in S$ such that for any $a, b, c \in \mathbb{R}$, $x \neq au + bv + cw$.

(b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

How to formulate '$\underbrace{\mathbf{u}, \mathbf{v}, \mathbf{w} \textit{ are linearly independent over } \mathbb{R}}_{(\text{\#})}$'?

Recall: '$H \to K$' is the same as '$(\sim H) \vee K$'.

- Formulation in words:

(#) reads: For any $a, b, c \in \mathbb{R}$, if $\underbrace{au + bv + cw = 0}_{H}$ then $\underbrace{(a=0 \text{ and } b=0 \text{ and } c=0)}_{K}$.

Equivalent formulation:

(#) reads: For any $a, b, c \in \mathbb{R}$, $[\underbrace{au+bv+cw \neq 0}_{(\sim H)} \underbrace{\text{or}}_{\vee} \underbrace{(a=0 \text{ and } b=0 \text{ and } c=0)}_{K}]$.

- Formulation in symbols:

(#) reads: $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})\{(au+bv+cw=0) \rightarrow [(a=0) \wedge (b=0) \wedge (c=0)]\}$

Equivalent formulation:

(#) reads: $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})\{(au+bv+cw \neq 0) \vee [(a=0) \wedge (b=0) \wedge (c=0)]\}$

How to formulate '$\underbrace{\mathbf{u}, \mathbf{v}, \mathbf{w} \textit{ are linearly dependent over } \mathbb{R}}_{\text{Negation of (\#)}}$'?

- Formulation in symbols:

Negation of (#) reads: $(\exists a \in \mathbb{R})(\exists b \in \mathbb{R})(\exists c \in \mathbb{R})\{(au+bv+cw=0) \wedge [(a \neq 0) \vee (b \neq 0) \vee (c \neq 0)]\}$

- Formulation in words:

Negation of (#) reads: There exist some $a, b, c \in \mathbb{R}$ such that $[\underbrace{au+bv+cw=0}_{H} \underbrace{\text{and}}_{\wedge} \underbrace{(a \neq 0 \text{ or } b \neq 0 \text{ or } c \neq 0)}_{(\sim K)}]$.

7. Examples from calculus of one variable.

(a) Let f be a real-valued function on $\mathbb{R}$, and $c \in \mathbb{R}$.

How to formulate '$\underbrace{f \textit{ attains a relative minimum at } c}_{(\spadesuit)}$'?

- Formulation in words:

(♠) reads: There exits some $\delta > 0$ such that for any $x \in \mathbb{R}$, (if $|x-c| < \delta$ then $f(x) \geq f(c)$).

[Equivalent formulation: There exists some $\delta > 0$ such that for any $x \in \mathbb{R}$, ($|x-c| \geq \delta$ or $f(x) \geq f(c)$).]

- Formulation in symbols:

(♠) reads: $(\exists \delta > 0)\{(\forall x \in \mathbb{R})\ [(|x-c| < \delta) \longrightarrow (f(x) \geq f(c))]\}$

Equivalent formulation: $(\exists \delta > 0)\{(\forall x \in \mathbb{R})[(|x-c| \geq \delta) \vee (f(x) \geq f(c))]\}$

How to formulate '$\underbrace{f \textit{ does not attain a relative minimum at } c}_{\text{Negation of } (\spadesuit)}$'?

- Formulation in symbols:

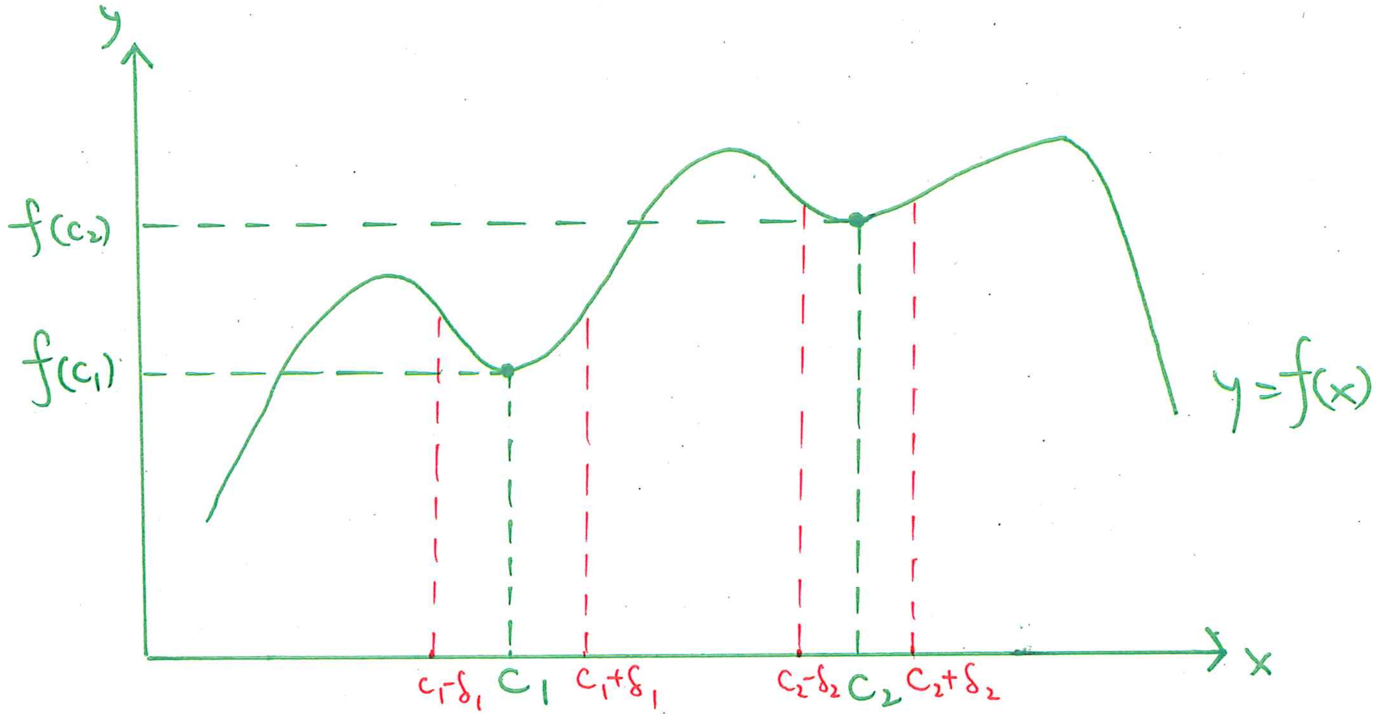
Negation of (♠) reads: $(\forall \delta > 0)\{(\exists x \in \mathbb{R})[(|x-c| < \delta) \wedge (f(x) < f(c))]\}$

- Formulation in words:

Negation of (♠) reads: For any $\delta > 0$, there exits some $x \in \mathbb{R}$ such that ($|x-c| < \delta$ and $f(x) < f(c)$).

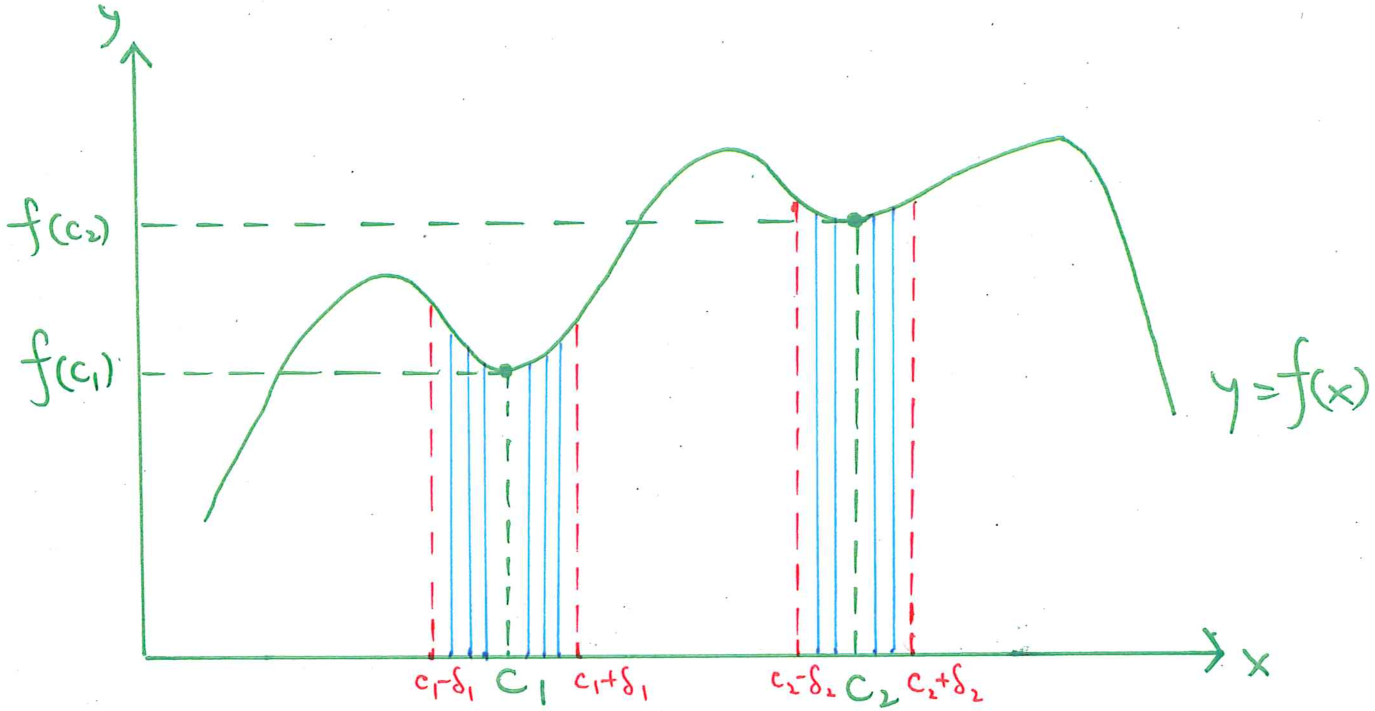
Let f be a function defined on $\mathbb{R}$, and $c \in \mathbb{R}$.

f attains a relative minimum at c. $\iff \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $f(x) \geq f(c)$.



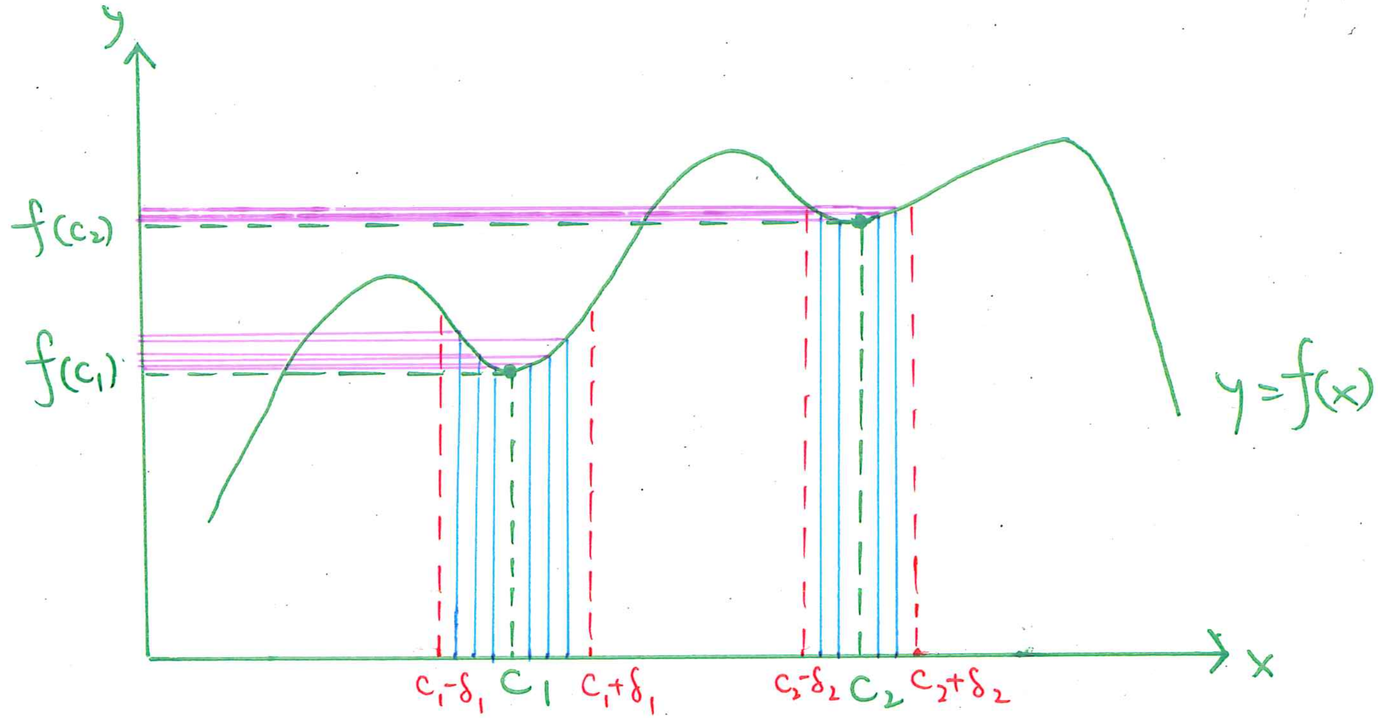
Let f be a function defined on $\mathbb{R}$, and $c \in \mathbb{R}$.

f attains a relative minimum at c. $\Leftrightarrow$ $\exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $f(x) \geq f(c)$.



Let f be a function defined on $\mathbb{R}$, and $c \in \mathbb{R}$.

f attains a relative minimum at c. $\Longleftrightarrow$ $\exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $f(x) \geq f(c)$.



(b) Let f be a real-valued function on $\mathbb{R}$, and $c \in \mathbb{R}$.

How to formulate '$\underbrace{f\ \textit{is continuous at}\ c}_{(\star)}$'?

- Formulation in words:

$(\star)$ reads: For any $\varepsilon > 0$, there exits some $\delta > 0$ such that for any $x \in \mathbb{R}$, (if $|x-c| < \delta$ then $|f(x)-f(c)| < \varepsilon$).

[Equivalent formulation: For any $\varepsilon > 0$, there exits some $\delta > 0$ such that for any $x \in \mathbb{R}$, ($|x-c| \geq \delta$ or $|f(x)-f(c)| < \varepsilon$).]

- Formulation in symbols:

$(\star)$ reads: $(\forall \varepsilon > 0)\{(\exists \delta > 0)\{(\forall x \in \mathbb{R})\,[(|x-c| < \delta) \longrightarrow (|f(x)-f(c)| < \varepsilon)]\}\}$

Equivalent formulation: $(\forall \varepsilon > 0)\{(\exists \delta > 0)\{(\forall x \in \mathbb{R})\,[(|x-c| \geq \delta) \vee (|f(x)-f(c)| < \varepsilon)]\}\}$

How to formulate '$\underbrace{f\ \textit{is not continuous at}\ c}_{\text{Negation of } (\star)}$'?

- Formulation in symbols:

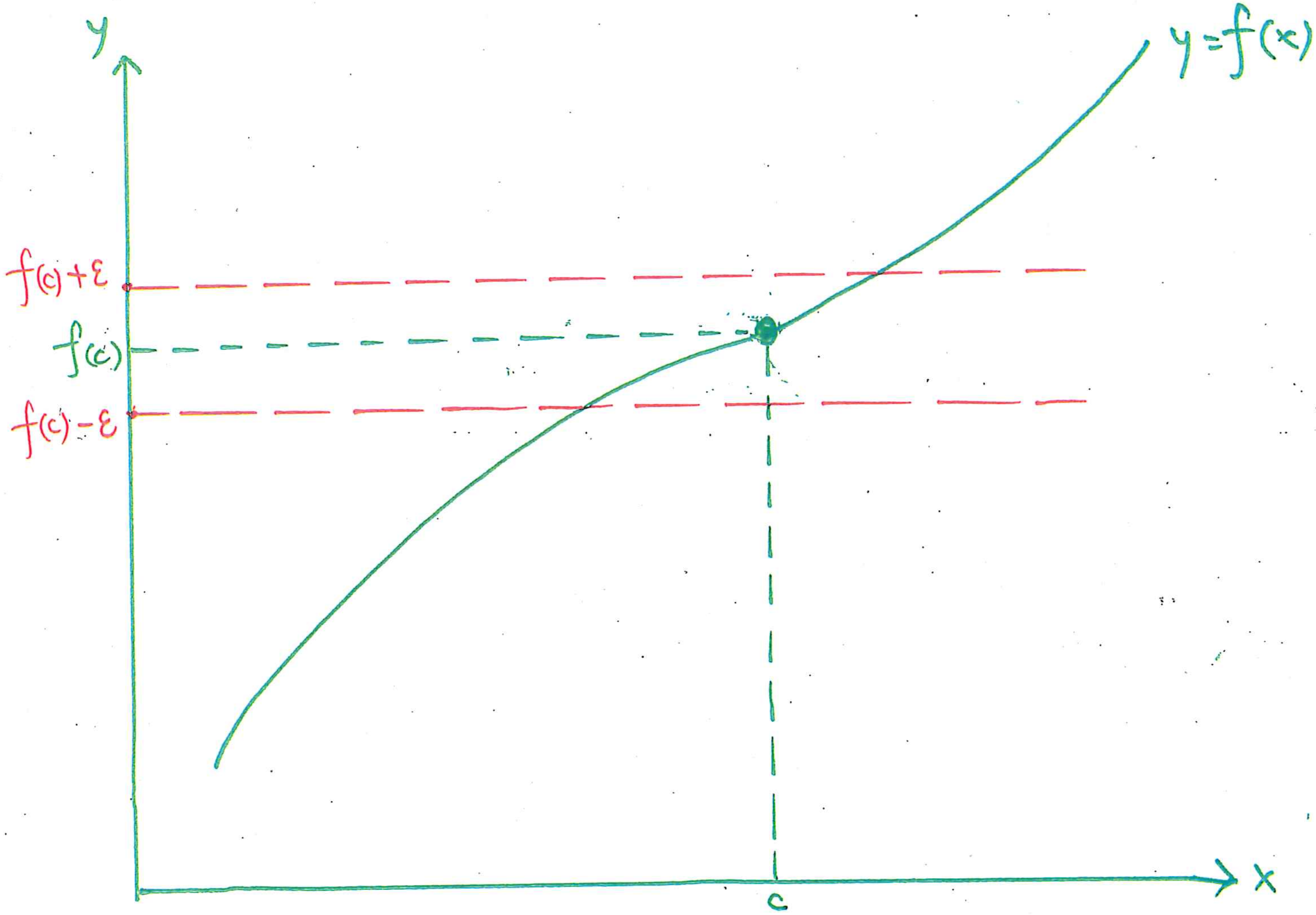
Negation of $(\star)$ reads: $(\exists \varepsilon > 0)\{(\forall \delta > 0)\{(\exists x \in \mathbb{R})\,[(|x-c| < \delta) \wedge (|f(x)-f(c)| \geq \varepsilon)]\}\}$

- Formulation in words:

Negation of $(\star)$ reads: There exits some $\varepsilon > 0$ such that for any $\delta > 0$, there exists some $x \in \mathbb{R}$ such that ($|x-c| < \delta$ and $|f(x)-f(c)| \geq \varepsilon$).

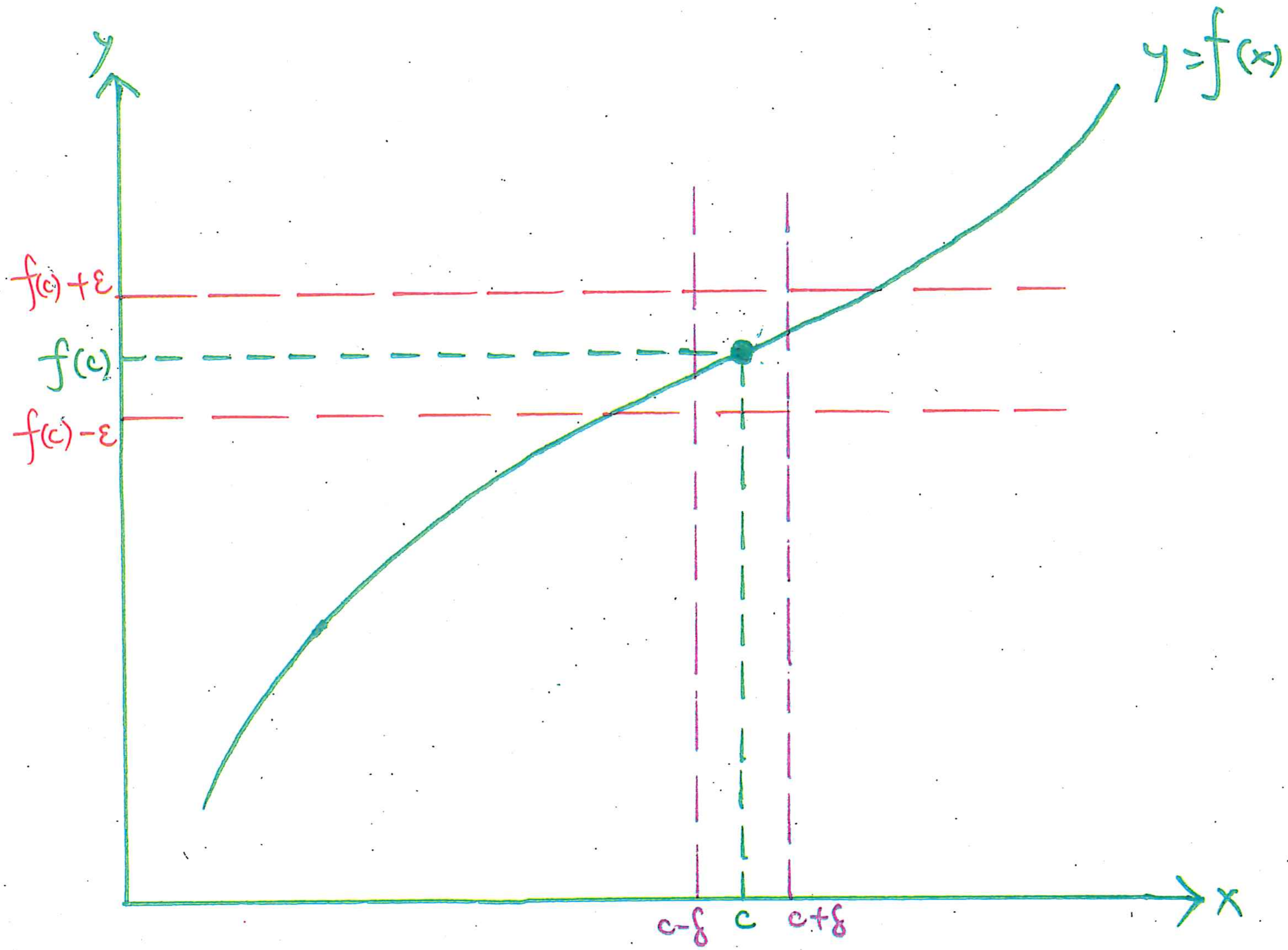
Let f be a function defined on $\mathbb{R}$, and $c \in \mathbb{R}$.

f is continuous at c. $\iff \forall \varepsilon > 0, \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|f(x)-f(c)| < \varepsilon$.



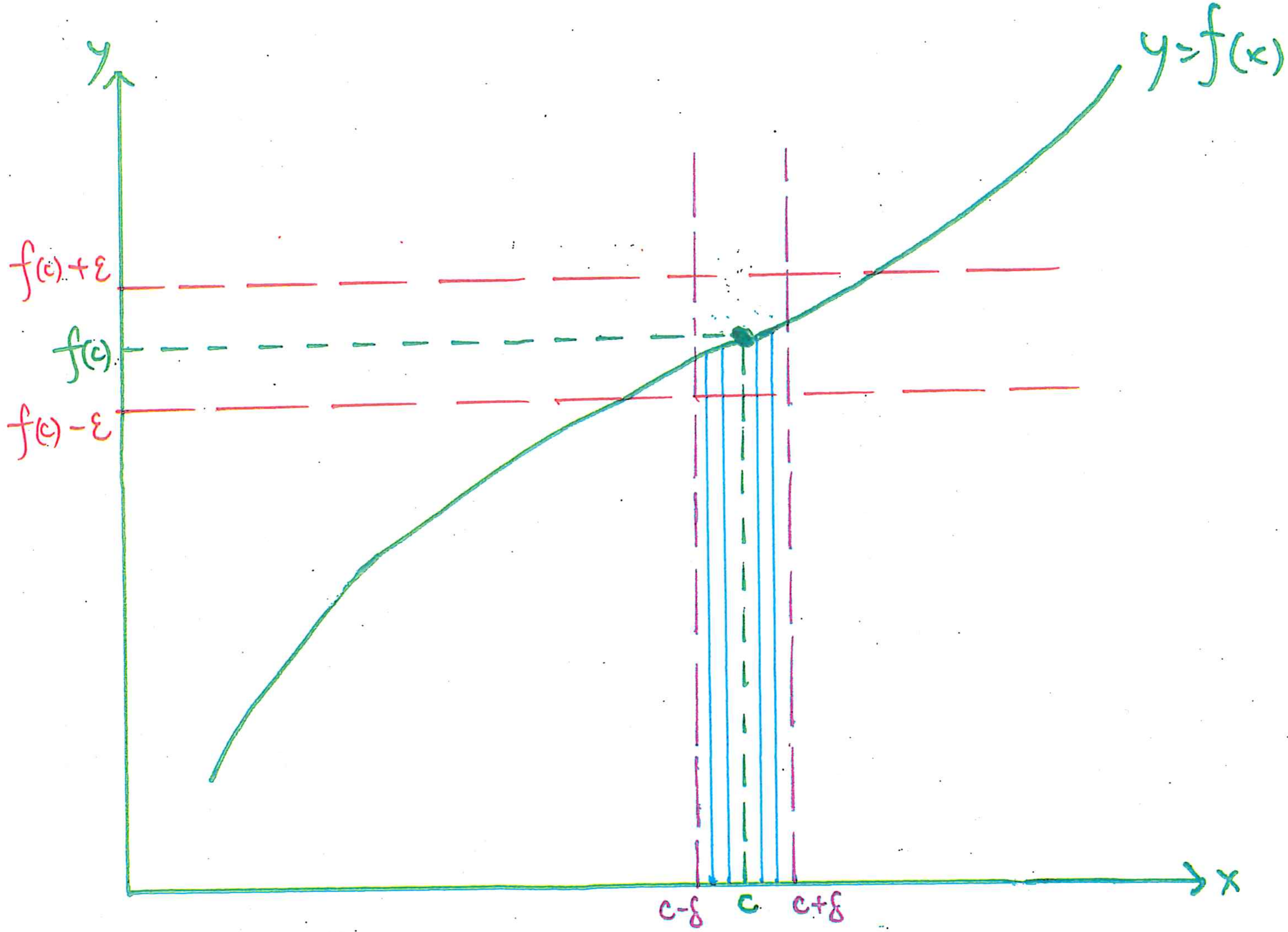
Let f be a function defined on $\mathbb{R}$, and $c \in \mathbb{R}$.

f is continuous at c. $\iff$ $\forall \varepsilon > 0, \exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.



Let f be a function defined on $\mathbb{R}$, and $c \in \mathbb{R}$.

f is continuous at c. $\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for any $x \in \mathbb{R}$, if $|x-c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.



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