- A (mathematical) statement is a sentence,
  or a number of carefully worded inter-related sentences,
  (with mathematical content,) for which it is meaningful to say
  it is true or it is false.

- Mathematical statements are 'predicates with no ambiguity'.

- A predicate with variables $x, y, z, \ldots$ is
  a statement 'modulo' the ambiguity of possibly one or several variables,
  so that, upon the specification of the variables by
  substituting concrete objects into the variables,
  a statement will be yielded.

Examples of statements.

(Some of them are true statements.
Some of them are false statements.)

ⓐ $1+1=2$.

ⓑ $1+1>3$.

ⓒ $\sqrt{2}$ is an irrational number.

ⓓ 6 is divisible by 8.

ⓔ There exits some natural number x such that x is divisible by 0.

ⓕ For any real number x, for any real number y, if $x<y$ then $x^2<y^2$.

ⓖ For any real number x, there exits some rational number y such that $|x-y|^2<|x-y|+1$.

ⓗ There exits some natural number y such that for any real number x, $y\leq|x|$.

Examples of predicates with some variables.

(Once the variables concerned are replaced by concrete objects, the respective predicates become statements.)

(a) $x + 1 = 2$. (a') $x + y = 2$. (a'') $x + 1 = y$.

(b) $x + y > 3$. (b') $x + y > z$.

(c) $\sqrt{x}$ is an irrational number.

(d) x is divisible by y.

(e) x is divisible by 0.

(e') There exits some natural number x such that x is divisible by y.

(f) For any real numby y, if $x < y$ then $x^2 < y^2$.

(f') If $x < y$ then $x^2 < y^2$.

(g) There exits some rational number y such that $|x - y|^2 < |x - y| + 1$.

(g') $|x - y|^2 < |x - y| + 1$.

(g'') For any real number x, there exits some rational number y such that $|x - y|^2 < |x - y| + z$.

The same statement may be presented as one 'very condensed' sentence or several inter-related sentences. The logical relation amongst the various 'components' of the statement is indicated by the logical connectives and quantifiers.

Illustration.

These are the same statement:

(a) For any real number x, for any real number y, if $x < y$ then $x^2 < y^2$.

(b) Let x, y be real numbers. Suppose $x < y$. Then $x^2 < y^2$.

(c) Let x, y be (arbitrary) objects. Suppose x, y are real numbers and $x < y$. Then $x^2 < y^2$.

(d) Let x, y be (arbitrary) objects. Suppose x is a real number and y is a real number and $x < y$. Then $x^2 < y^2$.

Words circled in red are where the presence of logical connectives are signified.

Words circled in blue are where the presence of quantifiers are signified.

- Predicate calculus is a study of the properties of the logical connectives and the quantifies.

- We want to know how the truth-hood and falsity of a 'compound statement/predicate' (formed by joining several 'shorter' statement/predicates with logical connectives and appending them with quantifiers) will be decided by the truth-hood and falsity of those 'shorter' statements/predicates in accordance with the logical connectives and quantifiers.

- Through the above, we look for some special types of statements, such as , tautology, contradiction, laws of logical equivalence, and rules of inference.

Five logical connectives

not	$\sim$	negation
and	$\wedge$	conjunction
or	$\vee$	disjunction
if ... then	$\rightarrow$	conditional
iff	$\leftrightarrow$	biconditional

- Every statement can be 'broken up' into (one or) several statements (or 'atoms') when the logical connectives in the statement are 'deleted'.
- 'Shorter' statements can be 'joint together' to give a 'longer' statement with the logical connectives. The resultant statement is called a compound statement.

Examples of usage

P : '2 is less than 3'

Q : '3 + 4 equals 8'.

$\sim P$: '2 is <ins>not</ins> less than 3'.
negation of P

$P \wedge Q$: '2 is less than 3 <ins>and</ins> 3+4 equals 8'.
conjunction of P, Q

$P \vee Q$: '2 is less than 3 <ins>or</ins> 3+4 equals 8'.
disjunction of P, Q

$P \rightarrow Q$: '<ins>If</ins> 2 is less than 3 <ins>then</ins> 3+4 equals 8'.
conditional from assumption P to conclusion Q

$P \leftrightarrow Q$: '2 is less than 3 <ins>iff</ins> 3+4 equals 8.'
biconditional from P to Q

Compound statements formed by joining P, Q with one logical connective.

Truth tables.

These are the simplest tools for studying the behaviour of the logical connectives.

'T' stands for 'true'
'F' stands for 'false'

P	$\sim P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical equivalence can be understood through truth tables.

After some work, we see that:

P	Q	$P \to Q$	$(\sim P) \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

This is one of several key features of the 'conditional' that deserve attention. They are to be found out.

Identical columns

Hence '$P \to Q$', '$(\sim P) \vee Q$' are logically equivalent.

P	Q	$P \leftrightarrow Q$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Identical columns

Examples

Tautology: a (compound) statement which is always true (irrespective of the truth values of the 'atoms').

$P \vee (\sim P)$

$P \rightarrow P$

$(P \wedge Q) \rightarrow P$

$P \rightarrow (P \vee Q)$

Contradiction: a (compound) statement which is always false (irrespective of the truth values of the 'atoms').

$P \wedge (\sim P)$

$P \leftrightarrow (\sim P)$

Contingent statement: a statement which is neither a tautology nor a contradiction.

$(P \vee Q) \rightarrow P$

$P \rightarrow (\sim P)$

Rules of Inference

Some of them are important and have names. They are to be found out.

Statements of the form

'blah-blah-blah' $\longrightarrow$ 'bleh-bleh-bleh'

$\underbrace{\text{'blah-blah-blah'}}_{\text{probably a compound statement}} \longrightarrow \underbrace{\text{'bleh-bleh-bleh'}}_{\text{probably a compound statement}}$

which are tautologies.

Examples of important rules of inference.

$$(P \wedge Q) \longrightarrow P$$

$$P \longrightarrow (P \vee Q)$$

$$[(P \to Q) \wedge (Q \to R)] \to (P \to R)$$

Modus ponens.

$$[(P \to Q) \wedge P] \longrightarrow Q$$

$$[(P \to Q) \wedge (\sim Q)] \to (\sim P)$$

...

Logical equivalence understood as tautologies.

Some are important and have names. They are to be found out.

Statements of the form

'blah-blah-blah' $\longleftrightarrow$ 'bleh-bleh-bleh'

which are tautologies.

('blah-blah-blah', 'bleh-bleh-bleh' are logically equivalent.)

Examples of important logical equivalence:

$$(P \to Q) \longleftrightarrow [(\sim P) \vee Q]$$

$$[(P \wedge Q) \vee R] \longleftrightarrow [(P \vee R) \wedge (Q \vee R)]$$

$$[\sim (P \wedge Q)] \longleftrightarrow (\sim P) \vee (\sim Q)$$

$$[\sim (\sim P)] \longleftrightarrow P$$

...