1. In mathematics very often we deal not with individual objects (say, individual numbers, individual vectors, individual functions), but collections of objects 'of the same type' (say, collections of integers, collections of real numbers, collections of vectors each with three real entries).

   In handling such collections, we use the language of sets.

   Remark.

   When coupled with a careful treatment of logic, this language can be developed into a mathematical theory, usually called *set theory*.

   The study of this theory is not our purpose here.

   We are mostly interested in making correct use of the language to help us do mathematics elsewhere.

2. **Undefined notions: 'set', 'belong to', 'element'.**

We begin with the notion

'*belong to*',

and define the notions

'*set*', '*element*'

in terms of it, by taking the three statements below to mean the same thing:

- '*The object x belongs to the set A.*'
- '*The object x is an element of the set A.*'
- '*The set A contains the object x as an element.*'

As for the notion '*belong to*', we leave it *unexplained.* At this level we allow heuristics to take over.

To save time, we write '$x \in A$' as a short-hand for

'*x belongs to A*'.

We write '$x \notin A$' as a short-hand for

'*x does not belong to A*'.

3. **Undefined notion: ‘equality’ for objects.**

We also leave the notion of ‘*equality*’ *unexplained*, and allow heuristics to take over.

To save time, we write ‘$x = y$’ as a short-hand for ‘x *equals* x’.

The key properties of the notion of ‘equality’ are stated below:

- *For any object* x, $x = x$.
- *For any objects* x, y, *if* $x = y$ *then* $y = x$.
- *For any objects* x, y, z, *if* $x = y$ *and* $y = z$ *then* $x = z$.

4. It may appear strange that something is left unexplained.

The 'problem' is that the notions

'*set*', '*belong to*', '*element*', and '*equality*'

remain *undefined*, and at the same time are *fundamental* in set language. (We are going to define everything else in set language in terms of these notions.)

This is deliberate. The reason is:

- To give the definition of an object, or a type of objects, is to specify what this object, or this type of objects, is amongst more general objects, or more general types of objects, which we have understood well enough.

  In other words, we 'anchor' the definition of what we want to explain via the definition upon something more general but well understood.

  In the situation of 'sets', we can 'anchor' upon nothing which is satisfactory.

  For example, we may 'define' a set as a collection. But we will then ask what we mean by 'collection'. We may 'define' a collection as an aggregate. But we will then ask what we mean by 'aggregate'. This will stop nowhere.

5. **Heuristic understanding of the notions of 'set equality', 'subset relation'**.

(a) Any two sets A, B are equal to each other as sets exactly when:

each of A, B contains as its elements every element of the other.

In this situation we write $A = B$.

(b) A 'relation' between two sets which is somewhat weaker than set equality is 'subset relation':

- Given any two sets A, B, the set A is a subset of the set B exactly when:

  every element of A belongs to B.

  In this situation we write $A \subset B$.
  (Some people's convention is '$A \subseteq B$'.)

Reminder: When $A = B$ holds, it will happen that $A \subset B$ also holds.

6. **'Small' sets**.

Suppose $a, b, \cdots, c$ are 'finitely many' objects (in the sense that we can list them out exhaustively).

Then we agree that we may present the set whose elements are exactly $a, b, \cdots, c$ as

$$\{a, b, \cdots, c\}.$$

We call this entity *the set whose elements are the objects* $a, b, \cdots, c$.

The symbols '$\{$','$\}$' in the notation '$\{a, b, \cdots, c\}$' are used for the purpose of reminding ourselves to think of this set (and read it) as an object on its own.

The symbol '$\{$' signifies the beginning of the list of objects which are elements of the set concerned.

The symbol '$\}$' signifies the end of this list.

7. **Conventions on the notations for 'small' sets.**

(a) 'Repetition in the list' does not count.

Provided an object, say, b, is an element of a 'small' set, say, S, the object b has to be presented at least once in the list representing S.

However, no matter how many more times b is presented, it still counts as once.

Illustration:

$\{a, a, b, c, b, b\}$ is the same set as $\{a, b, c\}$.

(b) 'Ordering in the list' does not matter.

Given any two lists, as long every object which is presented in each list is also presented in the other, the two lists will represent the same 'small' set, regardless of the order in which the objects is presented in each list.

Illustration:

$\{a, b, c\}$, $\{b, c, a\}$, $\{c, a, b\}$, $\{a, c, b\}$, $\{b, a, c\}$, $\{c, b, a\}$ all stand for the same set, namely the set whose elements are exactly the not necessarily distinct objects a, b, c.

8. **Empty set.**

The set to which no object belongs is denoted by $\emptyset$, and is called the empty set.

(Some people write $\{\ \}$ instead of $\emptyset$.)

9. **Examples.**

(a) The chain of symbols '$\{1, 3, 5, 7, 9\}$' stands for the set with exactly five elements, namely, $1, 3, 5, 7, 9$.

Denote this set by S.

i. It happens that $1 \in S$, $3 \in S$, $5 \in S$, $7 \in S$, $9 \in S$.

ii. It happens that $0 \notin S$, and $2 \notin S$.

iii. The sets below are all S itself in disguise:

$$\{1, 1, 3, 5, 7, 9\}, \quad \{1, 5, 3, 9, 7, 1\}, \quad \{9, 7, 5, 3, 1\}, \quad \{5, 5, 1, 3, 5, 7, 7, 9\}$$

iv. $\{1, 3, 5\}$ is a subset of S.

Reason: The elements of $\{1, 3, 5\}$ are exactly $1, 3, 5$. Each of $1, 3, 5$ is an element of S.

v. $\{5, 5, 7\}$ is a subset of S.

Reason: The elements of $\{5, 5, 7\}$ are exactly $5, 7$. Each of $5, 7$ is an element of S.

vi. $\{2, 3\}$ is not a subset of S.

Reason: 2 is an element of $\{2, 3\}$. It is not an element of S.

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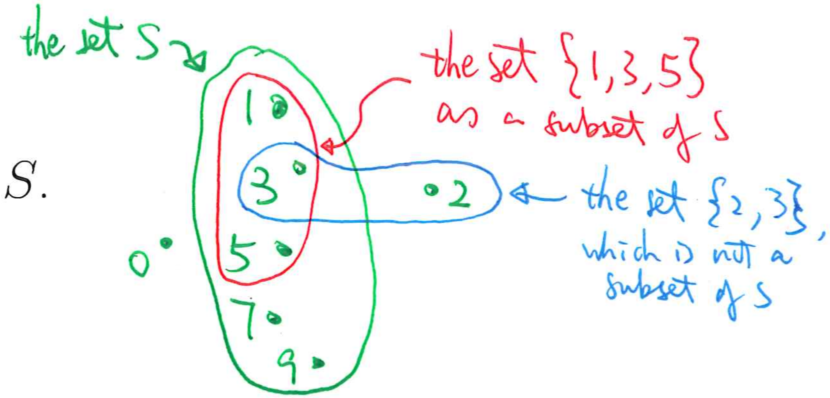

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(b) Note that $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,2,3\}$, $\{2,3\}$ are pairwise distinct sets. (Why?)

The chain of symbols '$\{\{1,2\},\{2,3\},\{1,3\}\}$' stands for the set with exactly three elements, namely, the three objects which are the sets $\{1,2\}$, $\{2,3\}$, $\{1,3\}$.

Denote this set by S.

i. It happens that $\{1,2\} \in S$, $\{2,3\} \in S$, $\{1,3\} \in S$.

ii. None of $\{1,2\}$, $\{2,3\}$, $\{1,3\}$ is the same as $\{1,4\}$. (Why?)
Then $\{1,4\} \notin S$.

iii. Take for granted that none of $\{1,2\}$, $\{2,3\}$, $\{1,3\}$ is a number.
Then $1 \notin S$.

iv. Note that $\{1,2,2,3\} = \{1,2,3\}$.
It is not the same as any one of $\{1,2\}$, $\{2,3\}$, $\{1,3\}$.
Then $\{1,2,2,3\} \notin S$.

v. $\{\{1,2\},\{2,3\}\}$ is a subset of S.
Reason: The elements of $\{\{1,2\},\{2,3\}\}$ are exactly $\{1,2\}$, $\{2,3\}$. Each of $\{1,2\}$, $\{2,3\}$ is an element of S.

vi. $\{\{1,3\}\}$ is a subset of S.
Reason: The only element of $\{\{1,3\}\}$ is $\{1,3\}$. It is an element of S.

vii. $\{1,3\}$ is not a subset of S.
Reason: 1 is an element of $\{1,3\}$. It is not an element of S.

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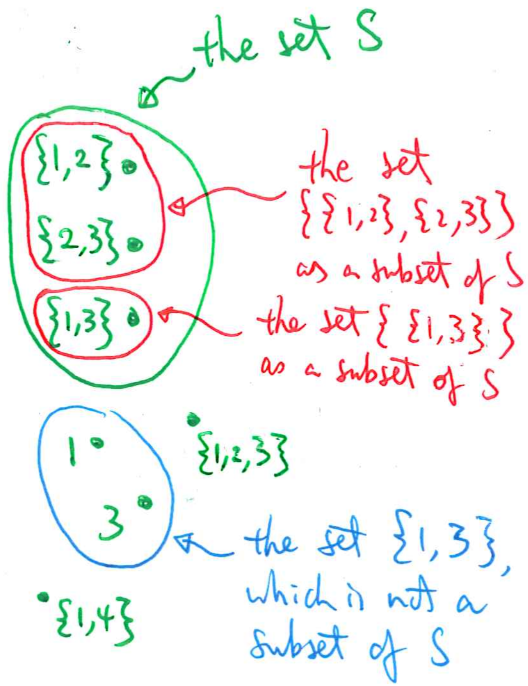

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