MATH1050 Examples: Miscellanies on inequalities.

1. Let n be a positive integer.

   (a) Prove that $k(n-k+1) \geq n$ for each integer k amongst $1, 2, \cdots, n$.

   (b) Hence, or otherwise, prove that $(n!)^2 \geq n^n$.

2. (a) Let $n \in \mathbb{N}\backslash\{0\}$. Prove that $\dfrac{2n}{2n+1} < \dfrac{2n+1}{2n+2}$.

   Remark. There is no need for mathematical induction.

   (b) Prove that $\displaystyle\prod_{k=1}^{5000} \frac{2k-1}{2k} < \frac{1}{100}$

3. (a) Prove that $\dfrac{2m-1}{2m} \leq \sqrt{\dfrac{3m-2}{3m+1}}$ for any $m \in \mathbb{N}\backslash\{0\}$.

   Remark. There is no need to use mathematical induction.

   (b) Hence, or otherwise, prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}} \quad \text{for any } n \in \mathbb{N}\backslash\{0\}$$

4. *In this question you may take for granted the validity of the statement* $(\star)$:

   $(\star)$ *Let u, v be positive real numbers. Suppose $u > v$. Then $\sqrt[3]{u} > \sqrt[3]{v}$.*

   (a) Prove the statement $(\sharp)$:

   $(\sharp)$ *Let x be a real number. Suppose $x > 0$. Then*

$$\sqrt[3]{x+2} - \sqrt[3]{x+1} < \frac{1}{3 \cdot \sqrt[3]{(x+1)^2}} < \sqrt[3]{x+1} - \sqrt[3]{x}.$$

   (b) Hence prove that $587 < \displaystyle\sum_{k=10}^{1000000} \frac{2}{\sqrt[3]{k^2}} < 588$.

   Remark. You may need the inequality $2.16^3 > 10$.

5. Prove the statement below $(\sharp)$:

   $(\sharp)$ *Let a, b be real numbers. Suppose $ab > 1$. Then $a^2 + 4b^2 > 4$.*

6. *Here we take for granted the validity of the statement* $(\sharp)$:

   $(\sharp)$ *Suppose $x, y \in \mathbb{R}$. Then $x^2 + y^2 \geq 2xy$. Moreover, equality holds iff $x = y$.*

   (a) Prove the statement $(\flat)$ below:

   $(\flat)$ *Suppose $u, v, w \in \mathbb{R}$. Then $u^2 + v^2 + w^2 \geq uv + vw + wu$. Moreover equality holds iff $u = v = w$.*

   (b) By applying the result described by $(\sharp)$, or otherwise, prove the statements below:

   i. *Suppose r, s, t be positive real numbers. Then $r + s + t \geq \sqrt{rs} + \sqrt{st} + \sqrt{tr}$.*

   ii. *Suppose $x, y, z \in \mathbb{R}$. Then $x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x + y + z)$.*

   iii. *Suppose a, b, c, d are positive real numbers. Then $(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) \geq 64(abcd)^{3/2}$.*

   iv. *Let p, q, r, s, t be positive real numbers. Suppose $pqrst = 1$. Then $(1+p)(1+q)(1+r)(1+s)(1+t) \geq 32$.*

7. (a) Prove the statement below:

   $(\sharp)$ *Suppose x, y are positive real numbers. Then $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$. Moreover, equality holds iff $x = y$.*

   (b) By applying the result described by $(\sharp)$, or otherwise, prove the statements below:

i. *Suppose* $a > 1$ *and* $b > 1$. *Then* $\log_a(b) + \log_b(a) \geq 2$. *Equality holds iff* $a = b$.

ii. *Suppose* $u \in \mathbb{R}$. *Then* $\dfrac{u^2+2}{\sqrt{u^2+1}} \geq 2$. *Equality holds iff* $u = 0$.

iii. *Suppose* $v \in \mathbb{R}$. *Then* $\dfrac{v^2}{1+v^4} \leq \dfrac{1}{2}$. *Equality holds iff* ($v = 1$ *or* $v = -1$).

8. We introduce this definition below:

   Let a, b, c *be three positive real numbers (not necessarily distinct from each other). The numbers* a, b, c *are said to* **constitute the three sides of a triangle** *if the three inequalities* $a + b > c$, $b + c > a$, $c + a > b$ *hold simultaneously.*

   (a) Let a, b be positive real numbers. Suppose $a \geq b$. Prove that there exists some positive real number c such that a, b, c constitute the three sides of a triangle.

   Remark. For the geometric interpretation, see Proposition 22, Book I of *Euclid's Elements*.

   (b) Let a, b, c be positive real numbers. Suppose a, b, c constitute the three sides of a triangle. Prove that $\sqrt{a}, \sqrt{b}, \sqrt{c}$ constitute the three sides of a triangle.

   (c) Let a, b, c be positive real numbers. Suppose a, b, c constitute the three sides of a triangle. Prove the statements below:

   i. $a^2 + b^2 + c^2 < 2(ab + bc + ca)$.

   ii. $3(ab + bc + ca) \leq (a + b + c)^2 < 4(ab + bc + ca)$.

   iii. $(a + b + c)(a + b - c) < 4ab$.

9. *In this question, you may assume without proof the validity of the statement* ($\sharp$)*:*

   ($\sharp$) *For any real numbers* μ, ν, *if* $0 < \mu < \nu < \dfrac{\pi}{2}$ *then* $0 < \sin(\mu) < \sin(\nu) < 1$.

   Let the angles at vertices A, B, C in $\triangle ABC$ be α, β, γ respectively. Suppose each angle in $\triangle ABC$ is an acute angle. Prove the statements below:

   (a) $\cos(\dfrac{\gamma}{2}) > \sin(\dfrac{\gamma}{2})$.

   (b) $\sin(\alpha) + \sin(\beta) > \cos(\alpha) + \cos(\beta)$.

   (c) $\sin(\alpha) + \sin(\beta) + \sin(\gamma) > \cos(\alpha) + \cos(\beta) + \cos(\gamma)$.

10. (a) Prove the statement ($\sharp$). (It may be easier to use the method of proof-by-contradiction.)

    ($\sharp$) *Let* a, b *be real numbers. Suppose* $|a| \leq 1$ *and* $|b| \leq 1$. *Then* $\sqrt{1-a^2} + \sqrt{1-b^2} \leq 2\sqrt{1 - \dfrac{(a+b)^2}{4}}$.

    (b) Applying the statement ($\sharp$), or otherwise, prove the statement ($\sharp\sharp$).

    ($\sharp\sharp$) *Let* a, b, c, d *be real numbers. Suppose* $|a| \leq 1$ *and* $|b| \leq 1$ *and* $|c| \leq 1$ *and* $|d| \leq 1$.

    Then $\sqrt{1-a^2} + \sqrt{1-b^2} + \sqrt{1-c^2} + \sqrt{1-d^2} \leq 4\sqrt{1 - \dfrac{(a+b+c+d)^2}{16}}$.

    (c) Applying the statement ($\sharp\sharp$), or otherwise, prove the statement ($\natural$).

    ($\natural$) *Let* a, b, c *be real numbers. Suppose* $|a| \leq 1$ *and* $|b| \leq 1$ *and* $|c| \leq 1$.

    Then $\sqrt{1-a^2} + \sqrt{1-b^2} + \sqrt{1-c^2} \leq 3\sqrt{1 - \dfrac{(a+b+c)^2}{9}}$.

    Remark. Can you generalize the results described above?

11. Prove the statement ($\sharp$). (It may be easier to use the method of proof-by-contradiction.)

    ($\sharp$) *Let* a, b *be real numbers. Suppose* $ab \neq 0$. *Then* $\left|\dfrac{a + \sqrt{a^2+2b^2}}{2b}\right| < 1$ *or* $\left|\dfrac{a - \sqrt{a^2+2b^2}}{2b}\right| < 1$.

12. Prove the statement ($\sharp$).

($\sharp$) *Let a, n be positive integers. Suppose $n \geq a$. Then* $(2a-1)^n + (2a)^n < (2a+1)^n$.

Remark. There is no need to apply mathematical induction.

13. Prove the statement ($\sharp$):

    ($\sharp$) *Suppose z is a non-zero complex number, and n is an integer greater than* 1.

    Then $\left(\dfrac{1-|z|^n+|z|^{2n}}{1+|z|^{2n}}\right)^{n^2} > 1 - \dfrac{n^2|z|^n}{1+|z|^{2n}}$.

    Remark. There is no need to apply mathematical induction. Bernoulli's Inequality may be relevant, however.

14. Let α be a complex number. Suppose $0 < |\alpha| < 1$. Define the number β by $\beta = \dfrac{1}{|\alpha|} - 1$. Note that $\beta > 0$.

    Let n be a positive integer.

    (a) Suppose $n \geq 3$. By applying the Binomial Theorem, or otherwise, prove that $(1+\beta)^n \geq \dfrac{n(n-1)}{2}\beta^2$.

    Hence deduce that $n|\alpha|^n \leq \dfrac{4}{n\beta^2}$.

    (b) Suppose $n \geq 4$. By applying the Binomial Theorem prove that $(1+\beta)^n \geq \dfrac{n(n-1)(n-2)}{6}\beta^3$.

    Hence deduce that $n^2|\alpha|^n \leq \dfrac{36}{n\beta^3}$.

    (c) Let k be a non-negative integer. Suppose $n \geq k+2$.

    By applying the Binomial Theorem, or otherwise, prove that $n^k|\alpha|^n \leq \dfrac{[(k+1)!]^2}{n\beta^{k+1}}$.

    Remark. The inequalities described here constitute the key step in the argument for the statement '$\lim\limits_{n\to\infty} n^k\alpha^n = 0$'.

15. For each $n \in \mathbb{N}\backslash\{0,1\}$, define $a_n = \sqrt[n]{n} - 1$.

    (a) Prove that $a_n \geq 0$ for any $n \in \mathbb{N}\backslash\{0,1\}$.

    (b) By applying the Binomial Theorem to the expression $(1+a_n)^n$, prove that $a_n \leq \sqrt{\dfrac{2}{n-1}}$ for any $n \in \mathbb{N}\backslash\{0,1\}$.

    Remark. The inequalities described here constitute the key step in the argument for the statement '$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$'.

16. Let $\alpha > 1$. For any $n \in \mathbb{N}\backslash\{0\}$, define $a_n = \alpha^{\frac{1}{n}} - 1$.

    (a) Prove that $a_n > 0$ for any $n \in \mathbb{N}\backslash\{0\}$.

    (b) By applying Bernoulli's Inequality, or otherwise, prove that $a_n \leq \dfrac{\alpha}{n}$ for any $n \in \mathbb{N}\backslash\{0\}$.

    Remark. The inequalities described here constitute the key step in the argument for the statement '$\lim\limits_{n\to\infty} \alpha^{1/n} = 1$'.

17. Prove the statement ($\sharp$):

    ($\sharp$) *Let m, n be integers greater than* 1. *Let ζ be a complex number. Suppose $0 < |\zeta| < \dfrac{1}{2}$. Then*

    $$\frac{1-|\zeta|-|\zeta|^m}{1-|\zeta|} < |(1+\zeta^m)(1+\zeta^{m+1}) \cdot \ldots \cdot (1+\zeta^{m+n})| < \frac{1-|\zeta|}{1-|\zeta|-|\zeta|^m}.$$

    Remark. You may find Weierstrass' Product Inequality useful.