0. With the help of the Bounded-Monotone Theorem and a basic result (Theorem (A)) on absolutely convergent infinite series (which you will learn in your *analysis* course), both stated below, we can 'extend' the Cauchy-Schwarz Inequality and Triangle Inequality to analogous results for 'square-summable infinite sequences in $\mathbb{R}$' (Theorem (B), Theorem (C) respectively).

1. **Definition.**
   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

   The infinite sequence $\left\{\sum_{j=0}^{n} a_j\right\}_{n=0}^{\infty}$ *is called the* **infinite series** *associated to the infinite sequence $\{a_n\}_{n=0}^{\infty}$.*

   For convenience, we usually denote the infinite sequence $\left\{\sum_{j=0}^{n} a_j\right\}_{n=0}^{\infty}$ *by* $\sum_{j=0}^{\infty} a_j$, *or by* $\sum a_j$.

   For each $k \in \mathbb{N}$, we refer to a_k as the k-th term of the infinite series $\sum_{j=0}^{\infty} a_j$.

   When the infinite sequence $\left\{\sum_{j=0}^{n} a_j\right\}_{n=0}^{\infty}$ *converges in $\mathbb{R}$, we may denote its limit by* $\sum_{n=0}^{\infty} a_n$.

   Warning. It may be confusing for beginners that the same symbols '$\sum_{j=0}^{\infty} a_j$' stand for two different objects: a specific infinite sequence which we call an infinite series, (in which the presence of the symbols '$\sum_{j=0}^{\infty}$', '∞' have nothing to do with convergence,) and the limit of that infinite sequence. But this is standard practice in any work on infinite series.

2. **Definition.**
   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

   (a) *The infinite series* $\sum_{j=0}^{\infty} a_j$ *is said to be* **absolutely convergent** *if the infinite series* $\sum_{j=0}^{\infty} |a_j|$ *is convergent.*

   (b) *The infinite sequence $\{a_n\}_{n=0}^{\infty}$ is said to be* **square-summable** *if the infinite series* $\sum_{j=0}^{\infty} {a_j}^2$ *is convergent.*

3. **Bounded-Monotone Theorem for increasing infinite sequences which are bounded above.**
   Let $\{u_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers. Suppose $\{u_n\}_{n=0}^{\infty}$ is increasing and is bounded above in $\mathbb{R}$. Then $\{u_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$, and its limit is the supremum of the set $\{x \in \mathbb{R} : x = u_n \text{ for some } n \in \mathbb{N}\}$. Furthermore, for any upper bound β of the infinite sequence $\{u_n\}_{n=0}^{\infty}$, the inequality $\lim_{n\to\infty} u_n \leq \beta$ holds.

   Also, for any $k \in \mathbb{N}$, the inequality $u_k \leq \lim_{n\to\infty} u_n$ holds.

4. **Theorem (A).**
   Let $\{v_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

   Suppose the infinite series $\sum_{j=0}^{\infty} v_j$ *is absolutely convergent. Then the infinite series* $\sum_{j=0}^{\infty} v_j$ *is convergent. Moreover the inequality* $\left|\sum_{n=0}^{\infty} v_n\right| \leq \sum_{n=0}^{\infty} |v_n|$ *holds. Equality holds iff the terms of $\{v_n\}_{n=0}^{\infty}$ are all non-negative or all non-positive.*

   Remark. This result is often expressed as: 'every absolutely convergent infinite series is convergent'.

5. **Proof of Theorem (A).**
   Let $\{v_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

Suppose the infinite series $\sum_{j=0}^{\infty} v_j$ is absolutely convergent.

For any $n \in \mathbb{N}$, we define $v_n^+ = \dfrac{|v_n| + v_n}{2}$ and $v_n^- = \dfrac{|v_n| - v_n}{2}$.

Note that, by definition, for any $n \in \mathbb{N}$, we have $|v_n| = v_n^+ + v_n^-$, $v_n = v_n^+ - v_n^-$, and $|v_n| \geq v_n^+ \geq 0$, $|v_n| \geq v_n^- \geq 0$, .

[We study the infinite series $\sum_{j=0}^{\infty} v_j^+$, $\sum_{j=0}^{\infty} v_j^-$. What are they really?

The infinite series $\sum_{j=0}^{\infty} v_j^+$ is the infinite series with all terms being non-negative, obtained from the infinite series $\sum_{j=0}^{\infty} v_j$ by replacing all its negative terms by 0.

The infinite series $\sum_{j=0}^{\infty} v_j^-$ is the infinite series with all terms being non-negative, obtained from the infinite series $\sum_{j=0}^{\infty} v_j$ by first replacing all its positive terms by 0 and then multiplying every term by -1.

So heuristically we expect '$\sum_{j=0}^{\infty} v_j = \sum_{j=0}^{\infty} v_j^+ - \sum_{j=0}^{\infty} v_j^-$' and '$\sum_{j=0}^{\infty} |v_j| = \sum_{j=0}^{\infty} v_j^+ + \sum_{j=0}^{\infty} v_j^-$'. However, there is the question of convergence.]

We verify that the infinite series $\sum_{j=0}^{\infty} v_j^+$ is convergent:

- For each $k \in \mathbb{N}$, $\sum_{j=0}^{k+1} v_j^+ - \sum_{j=0}^{k} v_j^+ = v_{k+1}^+ \geq 0$. Then the infinite sequence $\left\{ \sum_{j=0}^{n} v_j^+ \right\}_{n=0}^{\infty}$ is increasing.
- For each $k \in \mathbb{N}$, $\sum_{j=0}^{k} v_j^+ \leq \sum_{j=0}^{k} |v_j| \leq \sum_{n=0}^{\infty} |v_n|$. (Why does the second inequality hold?)

  Then the infinite sequence $\left\{ \sum_{j=0}^{n} v_j^+ \right\}_{n=0}^{\infty}$ is bounded above in $\mathbb{R}$, by $\sum_{n=0}^{\infty} |v_n|$.
- Hence, by the Bounded-Monotone Theorem, the infinite sequence $\left\{ \sum_{j=0}^{n} v_j^+ \right\}_{n=0}^{\infty}$ is convergent in $\mathbb{R}$.

Similarly we verify that the infinite series $\sum_{j=0}^{\infty} v_j^-$ is increasing and bounded above, and therefore convergent.

We observe that the limits $\sum_{j=0}^{\infty} v_j^+$, $\sum_{j=0}^{\infty} v_j^-$ are both non-negative because each term in the respective infinite series is non-negative.

Now we verify that the infinite series $\sum_{j=0}^{\infty} v_j$ is convergent , and the inequality $\left| \sum_{n=0}^{\infty} v_n \right| \leq \sum_{n=0}^{\infty} |v_n|$ holds:

- For any $k \in \mathbb{N}$, we have $\sum_{j=0}^{k} v_j = \sum_{j=0}^{k} (v_j^+ - v_j^-) = \sum_{j=0}^{k} v_j^+ - \sum_{j=0}^{k} v_j^-$.

  Then, since both infinite series $\sum_{j=0}^{\infty} v_j^+$, $\sum_{j=0}^{\infty} v_j^-$ are convergent, the infinite series $\sum_{j=0}^{\infty} v_j$ and is convergent.

  Moreover, the equality $\sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} v_n^+ - \sum_{n=0}^{\infty} v_n^-$ holds.

- For any $k \in \mathbb{N}$, we have $\displaystyle\sum_{j=0}^{k} |v_j| = \sum_{j=0}^{k} (v_j^+ + v_j^-) = \sum_{j=0}^{k} v_j^+ + \sum_{j=0}^{k} v_j^-$.

  Then since all three infinite series $\displaystyle\sum_{j=0}^{\infty} v_j^+$, $\displaystyle\sum_{j=0}^{\infty} v_j^-$, $\displaystyle\sum_{j=0}^{k} |v_j|$ are convergent, the infinite series $\displaystyle\sum_{j=0}^{\infty} v_j$, the equality $\displaystyle\sum_{n=0}^{\infty} |v_n| = \sum_{n=0}^{\infty} v_n^+ + \sum_{n=0}^{\infty} v_n^-$ holds.
- By the Triangle Inequality for real numbers, we have

$$\left|\sum_{n=0}^{\infty} v_n\right| = \left|\sum_{n=0}^{\infty} v_n^+ - \sum_{n=0}^{\infty} v_n^-\right| \leq \left|\sum_{n=0}^{\infty} v_n^+\right| + \left|\sum_{n=0}^{\infty} v_n^-\right| = \sum_{n=0}^{\infty} v_n^+ + \sum_{n=0}^{\infty} v_n^- = \sum_{n=0}^{\infty} |v_n|.$$

The argument for the necessary and sufficient conditions for the equality $\displaystyle\left|\sum_{n=0}^{\infty} v_n\right| = \sum_{n=0}^{\infty} |v_n|$ to hold is left as an exercise.

6. **Theorem (B). (Cauchy-Schwarz Inequality for 'square-summable infinite sequences in $\mathbb{R}$'.)**

   Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be infinite sequences of real numbers, neither of them being the zero sequence.

   Suppose $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are square-summable. Then the infinite series $\displaystyle\sum_{j=0}^{\infty} x_j y_j$ is absolutely convergent, and the statements below hold:

   (a) *The inequality* $\displaystyle\left|\sum_{n=0}^{\infty} x_n y_n\right| \leq \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$ *holds.*

   (b) *The statements $(\star_1)$, $(\star_2)$ are logically equivalent:*

   ($\star_1$) $\displaystyle\left|\sum_{n=0}^{\infty} x_n y_n\right| = \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.

   ($\star_2$) *There exist some $p, q \in \mathbb{R}$, not both zero, such that $px_j + qy_j = 0$ for any $j \in \mathbb{N}$. (The infinite sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are 'linearly dependent over $\mathbb{R}$'.)*

   Remark. In the context of the statement of Theorem (B), if one of the infinite sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ is the zero sequence, then the inequality in (a) trivially reduces to the equality in $(\star_1)$ of (b).

7. **Proof of Theorem (B).**

   Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be infinite sequences of real numbers, neither of them being the zero sequence.
   Suppose $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are square-summable.

   We verify that the infinite series $\displaystyle\sum_{j=0}^{\infty} x_j y_j$ is absolutely convergent:

   - The infinite sequence $\displaystyle\left\{\sum_{j=0}^{n} |x_j y_j|\right\}_{n=0}^{\infty}$ is increasing. (Why?)
   - For each $n \in \mathbb{N}$, by the Cauchy-Schwarz Inequality, the inequality $\displaystyle\sum_{j=0}^{n} |x_j y_j| \leq \left(\sum_{j=0}^{n} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{n} {y_j}^2\right)^{\frac{1}{2}}$ holds.

     Also, by assumption, the inequalities $\displaystyle\sum_{j=0}^{n} {x_j}^2 \leq \sum_{j=0}^{\infty} {x_j}^2$, $\displaystyle\sum_{j=0}^{n} {y_j}^2 \leq \sum_{j=0}^{\infty} {y_j}^2$ hold. (Why?)

     Therefore the infinite sequence $\displaystyle\left\{\sum_{j=0}^{n} |x_j y_j|\right\}_{n=0}^{\infty}$ is bounded above by $\displaystyle\left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.
   - Hence by the Bounded-Monotone Theorem, the infinite series $\displaystyle\sum_{j=0}^{\infty} |x_j y_j|$ is convergent.

     Moreover, the inequality $\displaystyle\sum_{n=0}^{\infty} |x_n y_n| \leq \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$ holds.

By definition, the infinite series $\sum_{j=0}^{\infty} x_j y_j$ is absolutely convergent.

(a) By Theorem (A), the infinite series $\sum_{j=0}^{\infty} x_j y_j$ is convergent, and the inequality $\left|\sum_{n=0}^{\infty} x_n y_n\right| \leq \sum_{n=0}^{\infty} |x_n y_n|$ holds.

Hence $\left|\sum_{n=0}^{\infty} x_n y_n\right| \leq \sum_{n=0}^{\infty} |x_n y_n| \leq \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.

(b) i. $[(\star_2) \Longrightarrow (\star_1)?]$

Suppose there exist some $p, q \in \mathbb{R}$, not both zero, such that $px_j + qy_j = 0$ for any $j \in \mathbb{N}$.
Without loss of generality, assume $p \neq 0$.

Then $\left|\sum_{n=0}^{\infty} x_n y_n\right| = \left|\sum_{n=0}^{\infty} -\frac{q}{p} \cdot {y_n}^2\right| = \frac{|q|}{|p|} \sum_{n=0}^{\infty} {y_n}^2 = \left(\sum_{n=0}^{\infty} \frac{|q|^2}{|p|^2} \cdot {y_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.

ii. $[(\star_1) \Longrightarrow (\star_2)?]$

Suppose $\left|\sum_{n=0}^{\infty} x_n y_n\right| = \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.

Then $\left|\sum_{n=0}^{\infty} x_n y_n\right| = \sum_{n=0}^{\infty} |x_n y_n|$, and $\sum_{n=0}^{\infty} |x_n y_n| = \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.

By the former, the terms of $\{x_n y_n\}_{n=0}^{\infty}$ are all non-negative or all non-positive.
Without loss of generality, assume the terms of $\{x_n y_n\}_{n=0}^{\infty}$ are all non-negative.

Then $\sum_{n=0}^{\infty} x_n y_n = \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$. Therefore $\left(\sum_{n=0}^{\infty} x_n y_n\right)^2 = \left(\sum_{n=0}^{\infty} {x_n}^2\right)\left(\sum_{n=0}^{\infty} {y_n}^2\right)$.

Define the polynomial $f(t)$ by $f(t) = \left(\sum_{n=0}^{\infty} {x_n}^2\right) t^2 + 2\left(\sum_{n=0}^{\infty} x_n y_n\right) t + \left(\sum_{n=0}^{\infty} {y_n}^2\right)$.

$f(t)$ is a quadratic polynomial with real coefficient. Its discriminant is 0. Then $f(t)$ has exactly one repeated real root, which we denote by r. We have

$$0 = f(r) = \left(\sum_{n=0}^{\infty} {x_n}^2\right) r^2 + 2\left(\sum_{n=0}^{\infty} x_n y_n\right) r + \left(\sum_{n=0}^{\infty} {y_n}^2\right) = \sum_{n=0}^{\infty} ({x_n}^2 r^2 + 2x_n y_n r + {y_n}^r) = \sum_{n=0}^{\infty} (x_n r + y_n)^2.$$

Then, for any $n \in \mathbb{N}$, we have $rx_n + 1 \cdot y_n = 0$.

8. **Theorem (C). (Triangle Inequality for 'square-summable infinite sequences in $\mathbb{R}$'.)**

Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be infinite sequences of real numbers, neither of them being the zero sequence.

Suppose $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are square-summable. Then the infinite sequence $\{x_n + y_n\}_{n=0}^{\infty}$ is square-summable, and the statements below hold:

(a) *The inequality* $\left[\sum_{n=0}^{\infty} (x_n + y_n)^2\right]^{\frac{1}{2}} \leq \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements $(*_1)$, $(*_2)$ are logically equivalent:*

$(*_1)$ $\left[\sum_{n=0}^{\infty} (x_n + y_n)^2\right]^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.

$(*_2)$ *There exist non-negative real numbers s, t, not both zero, such that $sx_j = ty_j$ for any $j \in \mathbb{N}$. (One of the infinite sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ is a non-negative scalar multiple of the other.)*

Remark. In the context of the statement of Theorem (C), if one of the infinite sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ is the zero sequence, then the inequality in (a) trivially reduces to the equality in $(*_1)$ of (b).

The proof of Theorem (C), as an application of Theorem (C), can be done in a similar way as the proof of the Triangle Inequality for real vectors as an application of the Cauchy-Schwarz Inequality for real vectors.