Theorem (6'). ('Weighted' version of Cauchy-Schwarz Inequality for definite integrals.)

Let a, b be real numbers, with $a<b$.

Let $m, f, g: [a,b] \to \mathbb{R}$ be functions.

Suppose $m(x) > 0$ for any $x \in [a,b]$.

Suppose neither f nor g is constant zero on $[a,b]$.

Suppose m, f, g are continuous on $[a,b]$. Then the statements below hold:

(a) The inequality

$$\left| \int_a^b f(u)g(u)\,m(u)\,du \right| \leq \left[\int_a^b (f(u))^2 m(u)\,du \right]^{\frac{1}{2}} \left[\int_a^b (g(u))^2 m(u)\,du \right]^{\frac{1}{2}}$$

holds.

(b) The statements $(\star_1)$, $(\star_2)$ are logically equivalent:

$(\star_1)$ $$\left| \int_a^b f(u)g(u)\,m(u)\,du \right| = \left[\int_a^b (f(u))^2 m(u)\,du \right]^{\frac{1}{2}} \left[\int_a^b (g(u))^2 m(u)\,du \right]^{\frac{1}{2}}.$$

$(\star_2)$ There exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that

$$pf(u) + qg(u) = 0 \quad \text{for any } u \in [a,b].$$

Theorem (7′). ('Weighted' version of Triangle Inequality for definite integrals.)

Let a, b be real numbers, with $a < b$.

Let $m, f, g : [a,b] \to \mathbb{R}$ be functions.

Suppose $m(x) > 0$ for any $x \in [a,b]$.

Suppose neither f nor g is constant zero on $[a,b]$.

Suppose m, f, g are continuous on $[a,b]$. Then the statements below hold:

(a) The inequality

$$\left[\int_a^b (f(u)+g(u))^2 m(u)\,du\right]^{\frac{1}{2}} \leq \left[\int_a^b (f(u))^2 m(u)\,du\right]^{\frac{1}{2}} \left[\int_a^b (g(u))^2 m(u)\,du\right]^{\frac{1}{2}}$$

holds.

(b) The statements $(*_1)$, $(*_2)$ are logically equivalent:

$(*_1)$ $$\left[\int_a^b (f(u)+g(u))^2 m(u)\,du\right]^{\frac{1}{2}} = \left[\int_a^b (f(u))^2 m(u)\,du\right]^{\frac{1}{2}} \left[\int_a^b (g(u))^2 m(u)\,du\right]^{\frac{1}{2}}$$

$(*_2)$ There exist some $s > 0$, $t > 0$ such that

$$s f(u) = t g(u) \text{ for any } u \in [a,b].$$

'Cauchy-Schwarz Inequality' for 'inner-product spaces'. (Glimpse of MATH2040/2048.)

Let V be a vector space over $\mathbb{R}$, with an inner product $\langle \cdot, \cdot \rangle$, and associated norm $\| \cdot \|$.

Suppose $x, y \in V$. Then:

(a) $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

(b) The statements $(☆_1)$, $(☆_2)$ are logically equivalent:

$(☆_1)$ $|\langle x, y \rangle| = \|x\| \cdot \|y\|$.

$(☆_2)$ x, y are linearly dependent over $\mathbb{R}$.

'Triangle Inequality' for 'inner-product spaces'. (Glimpse of MATH2040/2048.)

Let V be a vector space over $\mathbb{R}$, with an inner product $\langle \cdot, \cdot \rangle$, and associated norm $\| \cdot \|$.

Suppose $x, y \in V$. Then:

(a) $\| x + y \| \leq \| x \| + \| y \|$.

(b) The statements $(*_1)$, $(*_2)$ are logically equivalent:

$(*_1)$ $\| x + y \| = \| x \| + \| y \|$.

$(*_2)$ At least one of x, y is a non-negative scalar multiple of the other.

Ask: Can we generalize Theorem (3) from 'finite sums' to 'infinite sums'?

Theorem (3). (Cauchy-Schwarz Inequality for 'real vectors'.)

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $x_1, x_2, \cdots, x_n$ are not all zero and $y_1, y_2, \cdots, y_n$ are not all zero.

Then the statements below hold:

(a) *The inequality* $\left|\sum_{j=1}^{n} x_j y_j\right| \leq \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements $(\star_1)$, $(\star_2)$ are logically equivalent:*

$(\star_1)$ $\left|\sum_{j=1}^{n} x_j y_j\right| = \left(\sum_{j=1}^{n} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} {y_j}^2\right)^{\frac{1}{2}}$.

$(\star_2)$ *There exist some $p, q \in \mathbb{R}\backslash\{0\}$ such that $px_1 + qy_1 = 0$, $px_2 + qy_2 = 0$, ..., and $px_n + qy_n = 0$.*

Ask. Is Statement (☆) true?

Statement (☆). **(Cauchy-Schwarz Inequality for ~~'real vectors'~~ infinite sequences of real numbers)**

Let $\cancel{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n}$ $x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots$ $\in \mathbb{R}$. ($\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are infinite sequences of real numbers.)

Suppose $\cancel{x_1, x_2, \cdots, x_n}$ $x_0, x_1, x_2, \ldots$ are not all zero and $\cancel{y_1, y_2, \cdots, y_n}$ $y_0, y_1, y_2, \ldots$ are not all zero.

Then the statements below hold:

(a) *The inequality*
$$\left|\sum_{\cancel{j=1}\; j=0}^{\cancel{n}\;\infty} x_j y_j\right| \leq \left(\sum_{\cancel{j=1}\; j=0}^{\cancel{n}\;\infty} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{\cancel{j=1}\; j=0}^{\cancel{n}\;\infty} {y_j}^2\right)^{\frac{1}{2}}$$
holds.

(b) *The statements* $(\star_1)$, $(\star_2)$ *are logically equivalent:*

$(\star_1)$
$$\left|\sum_{\cancel{j=1}\; j=0}^{\cancel{n}\;\infty} x_j y_j\right| = \left(\sum_{\cancel{j=1}\; j=0}^{\cancel{n}\;\infty} {x_j}^2\right)^{\frac{1}{2}} \left(\sum_{\cancel{j=1}\; j=0}^{\cancel{n}\;\infty} {y_j}^2\right)^{\frac{1}{2}}.$$

$(\star_2)$ *There exist some* $p, q \in \mathbb{R}\backslash\{0\}$ *such that* $\cancel{px_1 + qy_1 = 0, px_2 + qy_2 = 0, \ldots, \text{ and } px_n + qy_n = 0.}$ $px_j + qy_j = 0$ for any $j \in \mathbb{N}$.

- Question.
  Do the expressions $\sum_{j=0}^{\infty} x_j y_j$, $\sum_{j=0}^{\infty} {x_j}^2$, $\sum_{j=0}^{\infty} {y_j}^2$ make sense as real numbers?
- Question clarified.
  Do the limits
  $\lim_{n\to\infty} \sum_{j=0}^{n} x_j y_j$,
  $\lim_{n\to\infty} \sum_{j=0}^{n} {x_j}^2$, $\lim_{n\to\infty} \sum_{j=0}^{n} {y_j}^2$
  exist?

Below are 'theoretical' tools for making sense of the expressions which appear in Statement (※).

Definition.

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

The infinite sequence $\left\{\sum_{j=0}^{n} a_j\right\}_{n=0}^{\infty}$ *is called the* **infinite series** *associated to the infinite sequence* $\{a_n\}_{n=0}^{\infty}$.

When the infinite sequence $\left\{\sum_{j=0}^{n} a_j\right\}_{n=0}^{\infty}$ *converges in* $\mathbb{R}$, *we may denote its limit by* $\sum_{n=0}^{\infty} a_n$.

Definition.

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of real numbers.

(a) *The infinite series* $\sum_{j=0}^{\infty} a_j$ *is said to be* **absolutely convergent** *if the infinite series* $\sum_{j=0}^{\infty} |a_j|$ *is convergent.*

(b) *The infinite sequence* $\{a_n\}_{n=0}^{\infty}$ *is said to be* **square-summable** *if the infinite series* $\sum_{j=0}^{\infty} {a_j}^2$ *is convergent.*

Theorem (B). (Cauchy-Schwarz Inequality for 'square-summable infinite sequences in $\mathbb{R}$'.)

Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be infinite sequences of real numbers, neither of them being the zero sequence.

Suppose $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are square-summable. [This ensures that it makes sense to talk about the limits $\lim_{n\to\infty}\sum_{j=0}^{n} x_j^2$, $\lim_{n\to\infty}\sum_{j=0}^{n} y_j^2$.]

Then the infinite series $\displaystyle\sum_{j=0}^{\infty} x_j y_j$ is absolutely convergent, and the statements below hold: [We are ensured that it makes sense to talk about the limit $\lim_{n\to\infty}\sum_{j=0}^{n} x_j y_j$.]

(a) *The inequality* $\displaystyle\left|\sum_{n=0}^{\infty} x_n y_n\right| \leq \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements $(\star_1)$, $(\star_2)$ are logically equivalent:*

$(\star_1)$ $\displaystyle\left|\sum_{n=0}^{\infty} x_n y_n\right| = \left(\sum_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}.$

$(\star_2)$ *There exist some $p, q \in \mathbb{R}$, not both zero, such that $px_j + qy_j = 0$ for any $j \in \mathbb{N}$.* *(The infinite sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are 'linearly dependent over $\mathbb{R}$'.)*

Ask: Can we generalize Theorem (4) from 'finite sums' to 'infinite sums'?

Theorem (4). (Triangle Inequality for 'real vectors'.)

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathbb{R}$.

Suppose $x_1, x_2, \cdots, x_n$ *are not all zero and* $y_1, y_2, \cdots, y_n$ *are not all zero.*

Then the statements below hold:

(a) *The inequality* $\left[\sum_{j=1}^{n}(x_j+y_j)^2\right]^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n}{x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n}{y_j}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements* $(*_1)$, $(*_2)$ *are logically equivalent:*

$$(*_1) \quad \left[\sum_{j=1}^{n}(x_j+y_j)^2\right]^{\frac{1}{2}} = \left(\sum_{j=1}^{n}{x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n}{y_j}^2\right)^{\frac{1}{2}}.$$

$(*_2)$ *There exist* $s > 0, t > 0$ *such that* $sx_1 = ty_1$, $sx_2 = ty_2$, ..., *and* $sx_n = ty_n$.

Ask. Is Statement (*) true?

Statement (*). (Triangle Inequality for ~~'real vectors'~~ infinite sequences of real numbers.)

Let $\cancel{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n}$ $x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots$ $\in \mathbb{R}$. ($\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are infinite sequences of real numbers.)

Suppose $\cancel{x_1, x_2, \cdots, x_n}$ $x_0, x_1, x_2, \dots$ are not all zero and $\cancel{y_1, y_2, \cdots, y_n}$ $y_0, y_1, y_2, \dots$ are not all zero.

Then the statements below hold:

(a) *The inequality*
$$\left[\sum_{\cancel{j=1}\ j=0}^{\cancel{n}\ \infty} (x_j + y_j)^2\right]^{\frac{1}{2}} \le \left(\sum_{\cancel{j=1}\ j=0}^{\cancel{n}\ \infty} {x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{\cancel{j=1}\ j=0}^{\cancel{n}\ \infty} {y_j}^2\right)^{\frac{1}{2}}$$
holds.

Question. Do the limits $\lim_{n\to\infty} \sum_{j=0}^{n} (x_j + y_j)^2$, $\lim_{n\to\infty} \sum_{j=0}^{n} {x_j}^2$, $\lim_{n\to\infty} \sum_{j=0}^{n} {y_j}^2$ exist?

(b) *The statements* $(*_1)$, $(*_2)$ *are logically equivalent:*

$(*_1)$
$$\left[\sum_{\cancel{j=1}\ j=0}^{\cancel{n}\ \infty} (x_j + y_j)^2\right]^{\frac{1}{2}} = \left(\sum_{\cancel{j=1}\ j=0}^{\cancel{n}\ \infty} {x_j}^2\right)^{\frac{1}{2}} + \left(\sum_{\cancel{j=1}\ j=0}^{\cancel{n}\ \infty} {y_j}^2\right)^{\frac{1}{2}}.$$

$(*_2)$ *There exist* $s > 0, t > 0$ *such that* $\cancel{sx_1 = ty_1, sx_2 = ty_2, \dots, \text{and } sx_n = ty_n}$ $sx_j = ty_j$ for any $j \in \mathbb{N}$.

Theorem (C). (Triangle Inequality for 'square-summable infinite sequences in $\mathbb{R}$'.)

Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be infinite sequences of real numbers, neither of them being the zero sequence.

Suppose $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are square-summable.

[This ensures that it makes sense to talk about the limits $\lim\limits_{n\to\infty}\sum\limits_{j=0}^{n} x_j^2$, $\lim\limits_{n\to\infty}\sum\limits_{j=0}^{n} y_j^2$.]

Then the infinite sequence $\{x_n + y_n\}_{n=0}^{\infty}$ is square-summable, and the statements below hold:

[We are ensured that it makes sense to talk about the limit $\lim\limits_{n\to\infty}\sum\limits_{j=0}^{n}(x_j+y_j)^2$.]

(a) *The inequality* $\left[\sum\limits_{n=0}^{\infty}(x_n+y_n)^2\right]^{\frac{1}{2}} \leq \left(\sum\limits_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} + \left(\sum\limits_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$ *holds.*

(b) *The statements $(*_1)$, $(*_2)$ are logically equivalent:*

$(*_1)$ $\left[\sum\limits_{n=0}^{\infty}(x_n+y_n)^2\right]^{\frac{1}{2}} = \left(\sum\limits_{n=0}^{\infty} {x_n}^2\right)^{\frac{1}{2}} + \left(\sum\limits_{n=0}^{\infty} {y_n}^2\right)^{\frac{1}{2}}$.

$(*_2)$ *There exist non-negative real numbers s, t, not both zero, such that $sx_j = ty_j$ for any $j \in \mathbb{N}$.*

(One of the infinite sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ is a non-negative scalar multiple of the other.)