MATH1050 Examples: Binomial coefficients and Binomial Theorem.

1. Let a be a real number, n be a positive integer, and $f(x)$ be the polynomial given by $f(x) = (1 + x + ax^2)^{6n}$. Denote the coefficients of the x-term, the x^2-term, and the x^3-term in the polynomial $f(x)$ by k_1, k_2, k_3 respectively.
   (a) Express k_1, k_2, k_3 in terms of a.
   (b) Suppose k_1, k_2, k_3 are in arithmetic progression.
      i. Prove that $a = \dfrac{An^2 + Bn + C}{9(2n-1)}$. Here A, B, C are some appropriate integers whose values you have to determine explicitly.
      ii. Further suppose $a \geq 0$. What is the value of n? Justify your answer.

2. Let n be a positive integer.
   (a) Suppose r is an integer amongst $0, 1, \cdots, n$. Prove that $\binom{n+1}{r+1} \Big/ \binom{n+1}{r} = \dfrac{n+1-r}{r+1}$.
   (b) Hence, or otherwise, deduce the equalities below:
      i. $\displaystyle\sum_{k=0}^{n} (k+1) \cdot \binom{n+1}{k+1} \Big/ \binom{n+1}{k} = \frac{An^2 + Bn + C}{2}$.
      ii. $\displaystyle\prod_{k=0}^{n} \left(\binom{n+1}{k+1} + \binom{n+1}{k} \right) = \frac{(n+D)^{n+E}}{[(n+F)!]} \cdot \left(\prod_{k=0}^{n} \binom{n+1}{k} \right)$.

   Here A, B, C, D, E, F are some positive integers whose respective values you have to determine explicitly.

3. Let m be a positive integer. Prove that $\displaystyle\sum_{k=0}^{m} 2^{2k} \binom{2m}{2k} = \frac{A^m + B}{2}$. Here A, B are some positive integers whose respective values you have to determine explicitly.

4. Prove the statement below, which is known as **Vandemonde's Theorem**:
   - *Let p, q, r be non-negative integers. Suppose $r \leq p + q$. Then* $\displaystyle\sum_{k=0}^{r} \binom{p}{k} \binom{q}{r-k} = \binom{p+q}{r}$.

   (*Hint.* Note that $(1+x)^{p+q} = (1+x)^p (1+x)^q$ as polynomials.)

5. Let n be a positive integer. Find the respective values of the numbers below. Leave your answer in terms of n.
   (a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2$.   (b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$.

   (*Hint.* Exploit the relation $\binom{n}{k} = \binom{n}{n-k}$.)

6. Let n be a positive integer. Find the respective value $\displaystyle\sum_{k=0}^{n} k \binom{n}{k}^2$. Leave your answer in terms of n.

7. Let n be a positive integer, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = (1+x)^n$ for any $x \in \mathbb{R}$.
   (a) Suppose $n \geq 3$.

      By differentiating f, or otherwise, prove that $\displaystyle\sum_{k=0}^{n} \frac{k(k-1)(k-2)}{3^k} \binom{n}{k} = \frac{n(n-1)(n-A) \cdot B^{n-C}}{3^n}$.

      Here A, B, C are some appropriate integers whose respective values you have to determine explicitly.

(b) By integrating f, or otherwise, prove that $\sum_{k=0}^{n} \frac{2^k}{(k+3)(k+2)(k+1)} \binom{n}{k} = \frac{A^{n+3} - 1 - 2(n+B)^2}{C(n+3)(n+2)(n+1)}$.

Here A, B, C are some appropriate integers whose respective values you have to determine explicitly.

8. (a) Let n, m be positive integers.

i. Verify the equality $x[(1+x)^n + (1+x)^{n+1} + \cdots + (1+x)^{n+m}] = (1+x)^{n+m+1} - (1+x)^n$ for polynomials.

ii. Let k be a positive integer. Write $c_{n,m,k} = \binom{n}{k} + \binom{n+1}{k} + \binom{n+2}{k} + \cdots + \binom{n+m}{k}$.

A. Suppose $k < n$. What is the value of $c_{n,m,k}$? Leave your answer in terms of n, m, k where appropriate.

B. Suppose $n \leq k \leq n+m$. What is the value of $c_{n,m,k}$? Leave your answer in terms of n, m, k where appropriate.

(b) Let m be a positive integer.

i. Applying the results in the previous parts, or otherwise, prove that

$$\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3) = 24(\binom{m+5}{5} - 1).$$

ii. Hence, or otherwise, find the value of $\sum_{r=0}^{m+4} r(r-1)(r-2)(r-3)$. Leave your answer in terms of m where appropriate.

9. Let p be a positive real number, satisfying $0 < p < 1$. Let n be a positive integer. For each $k = 0, 1, 2, \cdots, n$, define $a_k = \binom{n}{k} p^k (1-p)^{n-k}$.

(a) Show that $\sum_{r=0}^{n} a_r = 1$.

(b) Show that $0 < a_k < 1$ for each $k = 0, 1, \cdots, n$.

(c) Define $\mu = \sum_{r=0}^{n} r a_r$. Show that $\mu = np$.

(d) Further define $\sigma = \sqrt{\sum_{r=0}^{n} (r-\mu)^2 a_r}$.

i. Show that $\sigma^2 = \sum_{r=0}^{n} r^2 a_r - \mu^2$.

ii. Show that $\sum_{r=0}^{n} r(r-1) a_r = n(n-1)p^2$. Hence deduce that $\sigma^2 = np(1-p)$.

Remark. The finite sequence of numbers $a_0, a_1, \cdots, a_n$ gives a **binomial distribution**. The numbers μ, σ are the mean and the standard deviation for this distribution.