

MATH1050 Examples: Arithmetic progressions and geometric progressions.

1. Let a, b be non-zero numbers. Suppose $a, b, 1$ are in geometric progression and $a, b+1, 1$ are in arithmetic progression. Prove that $a, b+1, 2$ are in geometric progression.
2. Let a, b, c be real numbers. Suppose $c-a, c, c+b$ are non-zero, and are in geometric progression.
 - (a) Prove that $(Pa+Qb)c=ab$. Here P, Q are integers whose values you have to find explicitly.
 - (b) Suppose $a=b$. Find the value of a, b .
 - (c) Suppose $a \neq b$. Express c in terms of a, b .
3. Let a, b be non-zero real numbers. Suppose $a, \frac{1}{4}, b$ be in geometric progression, and $\frac{1}{a}, 5, \frac{1}{b}$ be in arithmetic progression. Find all possible values of a, b .
4. Let $a_0, a_1, a_2, a_3, \dots$ be a geometric progression of real numbers. Suppose $a_2 = -24$ and $a_5 = 81$ respectively.
 - (a) Find the value of a_0 .
 - (b) Prove that $a_n = \frac{(-1)^{n-P} B^{Qn-R}}{C^{n-S}}$ for each n . Here B, C, P, Q, R, S are some appropriate positive integers, independent of n , whose values you have to determine explicitly.
5. Let a be a non-zero real number. Suppose $-a, 2a, 3a^2, b$ are in geometric progression.
 - (a) Find the value of a .
 - (b) Find the value of b .
6. Let a, b, c, d be positive real numbers. Suppose a, b, c, d are in geometric progression. Prove that $\sqrt{\frac{a^5 + b^2c^2 + a^3c^2}{b^4c + d^4 + b^2cd^2}} = M \left(\frac{a}{b}\right)^N$. Here M, N are integers whose respective values you have to determined explicitly.
7. Let a_1, a_2, a_3, \dots be in arithmetic progression. Suppose $a_1 + a_2 + a_3 = 3$ and $a_1a_2a_3 = -24$. Find the value of a_n for each positive integer n .
8. Suppose the numbers a, b, c form an arithmetic progression.
 - (a) Suppose a, b, c also form a geometric progression. Prove that $a = b = c$.
 - (b) Now instead suppose a, c, b form a geometric progression. Further suppose that $a + b + c = 6$. Find the values of a, b, c respectively.
9. Let a, b, c be non-zero numbers. Suppose $a+b, b+c, c+a$ are all non-zero. Suppose $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in arithmetic progression. Prove that $\frac{bc}{b+c}, \frac{ca}{c+a}, \frac{ab}{a+b}$ are in arithmetic progression.
10. Let $u, v, w \in \mathbb{C} \setminus \{0\}$. Suppose they are pairwise distinct. Suppose $\frac{1}{u}, \frac{1}{v}, \frac{1}{w}$ form an arithmetic progression. Also suppose u, w, v form a geometric progression.
 - (a) Prove that $w = -2u$.
 - (b) Hence, or otherwise, prove that v, u, w form an arithmetic progression.
11. Let $a, b, c, d \in \mathbb{C} \setminus \{0\}$. Suppose $\frac{a}{b} = \frac{c}{d}$. Further suppose that a, b, c form an arithmetic progression. Prove that $\frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ form an arithmetic progression.

12. (a) Let $k \in \mathbb{N} \setminus \{0, 1\}$.

Suppose $b_0, b_1, b_2, \dots, b_k$ form arithmetic progression. Further suppose $b_0 + b_1 + b_2 + \dots + b_k = 0$.

Prove that $b_j + b_{k-j} = 0$ for each $j \in \llbracket 0, k \rrbracket$.

- (b) Let $\{c_p\}_{p=0}^\infty$ be an arithmetic progression.

Let $m, n \in \mathbb{N}$. Suppose $m < n$. Suppose $c_0 + c_1 + c_2 + \dots + c_m = c_0 + c_1 + c_2 + \dots + c_n$.

Prove that $c_0 + c_1 + c_2 + \dots + c_{m+n+1} = 0$.

13. Let $c_0, c_1, c_2, \dots, c_n \in \mathbb{C} \setminus \{0\}$. Suppose $c_0, c_1, c_2, \dots, c_n$ are in geometric progression.

Define $S = c_0 + c_1 + c_2 + \dots + c_n$, $P = c_0 \cdot c_1 \cdot c_2 \cdot \dots \cdot c_n$, and $R = \frac{1}{c_0} + \frac{1}{c_1} + \frac{1}{c_2} + \dots + \frac{1}{c_n}$.

Prove that $\left(\frac{S}{R}\right)^{n+1} = P^2$.

14. Let c be a positive real number. Let α, β, γ be real numbers, with $\alpha < \beta < \gamma$.

Suppose α, β, γ are in arithmetic progression. Further suppose $\sin(c\alpha) \neq 0$, and $\sin(c\alpha), \sin(c\beta), \sin(c\gamma)$ are in geometric progression.

Prove that the smallest possible value of $\beta - \alpha$ is $\frac{\pi}{c}$.

15. Let a_1, a_2, a_3, \dots be in arithmetic progression, with common difference d . Suppose $a_1 \neq 0$ and $d \neq 0$. For each positive integer n , denote by s_n the sum of a_1, a_2, \dots, a_n .

- (a) Prove that $\frac{s_{2n} - s_n}{s_n} = \frac{Aa_1 + (Bn - 1)d}{Aa_1 + (n - 1)d}$ for each positive integer n . Here A, B are positive integers whose values are independent of n , which you have to determine explicitly.

- (b) Suppose $\frac{s_4 - s_2}{s_2} = \frac{s_6 - s_3}{s_3}$.

i. Express d in terms of a_1 .

ii. Prove that $\frac{s_{2n} - s_n}{s_n}$ is constant for all values of n . What is the value of this constant?

16. (a) Let $m \in \mathbb{N}$. Let s be any number which is not equal to 1. Verify that

$$1 + 2s + 3s^2 + \dots + ms^{m-1} + (m+1)s^m = \frac{A - (m+2)s^{m+B} + (m+1)s^{m+C}}{(1-s)^D},$$

where A, B, C, D are integers independent of the value of s . You have to determine the respective values of A, B, C, D explicitly.

- (b) Let $n \in \mathbb{N}$. Let a, b be any real numbers. Prove that

$$(a-b)^E [a^m + 2a^{m-1}b + 3a^{m-2}b^2 + \dots + mab^{m-1} + (m+1)b^m] = a^{m+F} - (m+G)ab^{m+H} + (m+J)b^{m+K}.$$

Here E, F, G, H, J, K are integers independent of the value of m . You have to determine the respective values of E, F, G, H, J, K explicitly.