1a. Find $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}$.

Solution

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\frac{k}{n}}=\int_0^1\frac{1}{1+x}dx=\ln(1+x)|_{x=0}^{x=1}=\ln 2.$$

1b. Show that $\lim_{n\to\infty}\frac{1}{n}\{\sin\frac{t}{n}+\sin\frac{2t}{n}+\cdots+\sin\frac{(n-1)t}{n}\}=\frac{1-\cos t}{t}$.

Solution

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\{\sin\frac{t}{n}+\sin\frac{2t}{n}+\cdots+\sin\frac{nt}{n}\}&=\lim_{n\to\infty}\sum_{k=1}\frac{1}{n}\sin\frac{kt}{n}\\&=\int_0^1\sin(tx)dx\\&=-\frac{1}{t}\cos(tx)|_{x=0}^{x=1}=\frac{1-\cos t}{t}.\end{aligned}$$

2. If f is continuous on $[a,b]$, prove that there exists $\xi\in(a,b)$ such that $\int_a^b f(x)dx=(b-a)f(\xi)$.

Solution Define $F(x)=\int_a^x f(x)dx$, then $F'=f$ and $F(b)-F(a)=\int_a^b f(x)dx$. By mean value theorem, there exists $\xi\in(a,b)$ such that

$$\frac{F(b)-F(a)}{b-a}=F'(\xi)=f(\xi).$$

3a. Use the **definition** of **definite integrals** to show that

$$\left|\int_a^b f(x)dx\right|\leqslant\int_a^b|f(x)|dx.$$

Solution Because that $-|f(x)|\leqslant f(x)\leqslant|f(x)|$, hence

$$-\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k)|\Delta x_k\leqslant\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k)\Delta x_k\leqslant-\lim_{n\to\infty}\sum_{k=1}^{n}|f(x_k)|\Delta x_k.$$

By definition of definite integrals

$$-\int_a^b|f(x)|dx\leqslant\int_a^b f(x)dx\leqslant\int_a^b|f(x)|dx.$$

So

$$\left|\int_a^b f(x)dx\right|\leqslant\int_a^b|f(x)|dx.$$

3b. Use the above to show that

$$\lim_{n\to\infty}\int_0^{2\pi}\frac{\sin(nx)}{x^2+n^2}dx=0.$$

Solution

$$\begin{aligned}|\int_0^{2\pi}\frac{\sin(nx)}{x^2+n^2}dx| &\leqslant \int_0^{2\pi}|\frac{\sin(nx)}{x^2+n^2}|dx\\ &\leqslant \int_0^{2\pi}\frac{1}{x^2+n^2}dx\\ &\leqslant \int_0^{2\pi}\frac{1}{n^2}dx=\frac{2\pi}{n^2}\end{aligned}$$

By squeeze theorem, the limit holds.

3c. *Solution*

$$\begin{aligned}\int_1^2\frac{dx}{(x^2-2x+4)^{3/2}} &= \int_1^2\frac{dx}{[(x-1)^2+3]^{3/2}}\\ &\xlongequal{\text{let }(x-1)=\sqrt{3}\tan u} \int_0^{\pi/6}\frac{\sqrt{3}\sec^2 u\,du}{(3+3\tan^2 u)^{3/2}}\\ &= \int_0^{\pi/6}\frac{\sqrt{3}\sec^2 u\,du}{[3\sec^2 u]^{3/2}}=\frac{1}{3}\int_0^{\pi/6}\cos u\,du=\frac{1}{3}\sin u|_0^{\pi/6}=\frac{1}{6}.\end{aligned}$$

3d. *Solution*

$$\begin{aligned}\int x\ln(x+3)dx &= \frac{1}{2}\int\ln(x+3)d(x^2)\\ &= \frac{1}{2}[x^2\ln(x+3)-\int x^2 d(\ln(x+3))]\\ &= \frac{1}{2}[x^2\ln(x+3)-\int\frac{x^2}{x+3}dx]\\ &= \frac{1}{2}[x^2\ln(x+3)-\int(x-3+\frac{9}{x+3})dx]\\ &= \frac{1}{2}x^2\ln(x+3)-\frac{1}{2}(\frac{1}{2}x^2-3x+9\ln(x+3))+c.\end{aligned}$$

So $\int_0^1 x\ln(x+3)dx=\frac{5}{4}-4\ln 4+9\ln 3/2$.

3e. Find $\int_0^\pi\frac{x\sin x}{1+\cos^2 x}dx$ (hint: try $x=\pi-y$).'

Solution Let $x=\pi-y$, $I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$, so

$$\begin{aligned}I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx=&\int_0^\pi \frac{(\pi-y)\sin y}{1+\cos^2 y}dy\\ =&\pi\int_0^\pi \frac{\sin y dy}{1+\cos^2 y}-\int_0^\pi \frac{y\sin y}{1+\cos^2 y}dy\\ =&\pi\int_0^\pi \frac{d(\cos y)}{1+\cos^2 y}-I\\ =&\pi\arctan(\cos y)|_0^\pi - I=\pi^2/2-I.\end{aligned}$$

So $I=\pi^2/2$.

4. More **indefinite integral** exercises:

4a. $\displaystyle\int \frac{dx}{1-\cos x}$

Solution

$$\int \frac{dx}{1-\cos x}=\int \frac{1}{2\sin^2\frac{x}{2}}dx=\int \frac{1}{\sin^2\frac{x}{2}}d\left(\frac{x}{2}\right)=-\cot\frac{x}{2}+c.$$

4b. $\displaystyle\int \tan^5 x\, dx$

Solution

$$\begin{aligned}\int \tan^5 x\,dx=&\int \frac{\sin^4 x\sin x\,dx}{\cos^5 x}\\ =&-\int \frac{(1-\cos^2 x)^2 d(\cos x)}{\cos^5 x}\\ \overset{\text{let } t=\cos x}{=\!=\!=\!=}&-\int \frac{(1-t^2)^2 dt}{t^5}\\ =&-\int (t^{-5}-2t^{-3}+t^{-1})dt\\ =&\frac{1}{4}t^{-4}-t^{-2}-\log|t|+c\\ =&\frac{1}{4}(\cos x)^{-4}-(\cos x)^{-2}-\log|\cos x|+c.\end{aligned}$$

4c. $\displaystyle\int \cos^5 x\sin^3 x\,dx$

Solution

$$\begin{aligned}\int \cos^5 x\sin^3 x\,dx=&\int (1-\sin^2 x)^2\sin^3 x\,d(\sin x)\\ \overset{\text{let } t=\sin x}{=\!=\!=\!=}&\int (t^3-2t^5+t^7)\,dt\\ =&\frac{1}{4}t^4-\frac{1}{3}t^6+\frac{1}{8}t^8+c\\ =&\frac{1}{4}\sin^4 x-\frac{1}{3}\sin^6 x+\frac{1}{8}\sin^8 x+c.\end{aligned}$$

4d. $\int \frac{dx}{\cos x \sin^2 x}$

Solution

$$\begin{aligned}
\int \frac{dx}{\cos x \sin^2 x} &= \int \frac{\cos x\, dx}{\cos^2 x \sin^2 x} \\
&= \int \frac{d \sin x}{(1-\sin^2 x)\sin^2 x} \\
&\xlongequal{\text{let } t=\sin x} \int (\frac{1}{t^2} + \frac{1}{2\,(t+1)} - \frac{1}{2\,(t-1)}) dt \\
&= -\frac{1}{t} - \frac{1}{2}\log(1-t) + \frac{1}{2}\log(t+1) + c \\
&= -\csc(x) - \log\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) + \log\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) + c
\end{aligned}$$

4e. $\int \frac{dx}{(1-x^2)^{3/2}}$

Solution

$$\begin{aligned}
\int \frac{dx}{(1-x^2)^{3/2}} &\xlongequal{x=\cos t,\, t=\arccos x} \int \frac{d\cos t}{\sin^3 t} \\
&= -\int \frac{1}{\sin^2 t} dt = -\cot t + c \\
&= \frac{x}{\sqrt{1-x^2}} + c
\end{aligned}$$

4f. $\int x^2 \sqrt{16-x^2}\, dx$

Solution

$$\begin{aligned}
\int x^2\sqrt{16-x^2}\, dx &\xlongequal{x=4\sin t,\, t\in[-\pi/2,\pi/2]} \int 16 \sin^2 t\, 4\cos t\, d\sin t \\
&= 64 \int \sin^2 t \cos^2 t\, dt \\
&= 16 \int \sin^2(2t) dt \\
&= 8 \int (1-\cos 4t) dt = 8t - 2\sin 4t + c \\
&= \frac{1}{4} x\sqrt{16-x^2}\,(x^2-8) + 32\sin^{-1}\left(\frac{x}{4}\right) + c
\end{aligned}$$

4g. $\int \frac{dx}{(4x^2+1)^{3/2}}$

Solution

$$\int \frac{dx}{(4x^2+1)^{3/2}} \overset{x=\tan s/2}{=\!=\!=} \int \frac{d\,(\tan s/2)}{\sec^3 t}$$
$$= \int \frac{1}{2}\cos s\,ds = \frac{1}{2}\sin s + c = \frac{x}{\sqrt{4\,x^2+1}} + c$$

4h. $\displaystyle\int \frac{dx}{(2x-x^2)^{3/2}}$

Solution

$$\int \frac{dx}{(2x-x^2)^{3/2}} = \int \frac{1}{[1-(x-1)^2]^{3/2}} d\,(x-1)$$
$$=\text{refer to Problem 4e.}$$

4i. $I_n = \displaystyle\int \frac{1}{\sin^n x} dx$, show that $I_n = -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1} I_{n-2}, n \geqslant 2.$

Solution

$$I_n = -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1} I_{n-2}, n \geqslant 2.$$

$$\begin{aligned}
I_n &= \int \frac{\cos^2 x + \sin^2 x}{\sin^n x} dx \\
&= \int \frac{\cos x}{\sin^n x} d(\sin x) + I_{n-2} \\
&= \frac{\cos x}{\sin^{n-1} x} - \int \sin x\,(\frac{\cos x}{\sin^n x})' dx + I_{n-2} \\
&= \frac{\cos x}{\sin^{n-1} x} + \int \frac{n\cos^2 x + \sin^2 x}{\sin^n x} dx + I_{n-2} \\
&= \frac{\cos x}{\sin^{n-1} x} + n I_n + (2-n) I_{n-2}
\end{aligned}$$

hence the result.