THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2017-2018
Midterm Examination

Name (in print): ____

Student ID: ________________ Programme: ____________ Section: MATH1010____

* * *

INSTRUCTIONS to students:

1. The examination lasts 90 minutes.
2. There are 6 problems, worth a total of 100 points.
3. Answer all questions. Show work to justify all answers.
4. Answer the questions in the space provided.

* * *

FOR MARKERS' USE ONLY:

1	
2	
3	
4	
5	
6	
Total	/100 points

1. (20 marks) Find $\frac{dy}{dx}$ where:

(a) $y = \frac{x^4 + 5x}{1 - e^x}$

(b) $y = \sin\left(\sqrt{x \ln x}\right)$

(c) $y \sin x + x \cos y = 1$

(d) $x^y = y, \quad x > 0$

2. (15 marks) Evaluate the following limits.

(a) $\lim_{x \to 0^-} \frac{\sin x}{|x| + 4x}$

(b) $\lim_{x \to +\infty} \frac{\sin x}{x}$

(c) $\lim_{x \to +\infty} (x - \sqrt{x^2 + 1})$

3. (15 marks) Let a_n be the sequence defined by

$$\begin{cases} a_{n+1} = 1 - (a_n - 1)^2, \text{ for } n \geq 1 \\ a_1 = \dfrac{1}{100}. \end{cases}$$

(a) Show that $0 \leq a_n \leq 1$ for any $n \geq 1$.

Solution:

(b) Show that $a_{n+1} - a_n \geq 0$ for any $n \geq 1$.

Solution:

(c) Explain whether the limit of a_n exists and find the limit if it exists.

Solution:

4. (20 marks) Let n be a positive integer. Let:

$$f(x) = \begin{cases} x^3, & \text{if } x < 0; \\ x^n, & \text{if } x \geq 0. \end{cases}$$

(a) Find $f'(x)$ for $x > 0$.

Solution:

(b) Find all positive integers n such that:

i. $f'(0)$ exists.

ii. $f''(0)$ exists.

(Recall that by definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.)

Justify your answers.

Solution:

5. (15 marks) Let f be a function which is continuous on $[a, b]$, differentiable in (a, b) and satisfies $f(a) = f(b) = 0$. By considering the function $e^{x/s} f(x)$, show that for any non-zero real number s there exists $d \in (a, b)$ satisfying

$$sf'(d) + f(d) = 0.$$

Solution:

6. (15 marks) Determine whether there is any function satisfying all of the following conditions:

   (i) f is differentiable in $(0, 2)$.

   (ii) f is continuous on $[0, 2]$.

   (iii) f satisfies $f(0) = 1, f(2) = 10, f'(x) \leq 2,$ for each $x \in (0, 2)$.

   Give an example of such a function if you think "yes", or explain why no such functions exist if you think "no".

   Solution: