THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2017-2018
Midterm Examination

Name (in print): ____

Student ID: _______________ Programme: ___________ Section: MATH1010____

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INSTRUCTIONS to students:

1. The examination lasts 90 minutes.
2. There are 6 problems, worth a total of 100 points.
3. Answer all questions. <u>Show work to justify all answers.</u>
4. Answer the questions in the space provided.

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FOR MARKERS' USE ONLY:

1	
2	
3	
4	
5	
6	
Total	/100 points

1. (20 marks) Find $\dfrac{dy}{dx}$ where:

(a) $y = \dfrac{x^4 + 5x}{1 - e^x}$

$$\frac{dy}{dx} = \frac{(4x^3 + 5)(1 - e^x) - (-e^x)(x^4 + 5x)}{(1 - e^x)^2}$$

$$= \frac{4x^3 + 5 - 4x^3e^x - 5e^x + x^4e^x + 5xe^x}{(1 - e^x)^2}$$

$$= \frac{e^x(x^4 - 4x^3 + 5x - 5) + 4x^3 + 5}{(1 - e^x)^2}.$$

(b) $y = \sin\left(\sqrt{x \ln x}\right)$

$$\frac{dy}{dx} = \cos\left(\sqrt{x \ln x}\right) \frac{1}{2\sqrt{x \ln x}}(\ln x + 1)$$

$$= \frac{\cos\left(\sqrt{x \ln x}\right)(\ln x + 1)}{2\sqrt{x \ln x}}.$$

(c) $y \sin x + x \cos y = 1$

$$\frac{dy}{dx} \sin x + y \cos x + \cos y - x \sin y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y \cos x + \cos y}{x \sin y - \sin x}.$$

(d) $x^y = y, \quad x > 0$

Obiviously $y > 0$. Since $y \ln x = \ln y$,

$$\frac{dy}{dx} \ln x + \frac{y}{x} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{y}{x} = (\frac{1}{y} - \ln x) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y/x}{(1 - y \ln x)/y}$$

$$\frac{dy}{dx} = \frac{y^2}{x - xy \ln x}$$

or $\dfrac{dy}{dx} = \dfrac{y^2}{x - x \ln y}$.

2. (15 marks) Evaluate the following limits.

(a) $\lim_{x\to 0^-} \frac{\sin x}{|x| + 4x}$

$$= \lim_{x\to 0^-} \frac{\sin x}{-x + 4x}$$

$$= \lim_{x\to 0^-} \frac{\sin x}{3x}$$

$$= \lim_{x\to 0^-} \frac{1}{3}\frac{\sin x}{x}$$

$$= \frac{1}{3} \lim_{x\to 0^-} \frac{\sin x}{x}$$

$$= \frac{1}{3}.$$

(b) $\lim_{x\to +\infty} \frac{\sin x}{x}$

Since $-1 \leqslant \sin x \leqslant 1$,

$$\frac{-1}{x} \leqslant \frac{\sin x}{x} \leqslant \frac{1}{x} \text{ for } x > 0.$$

$$\lim_{x\to +\infty} \frac{-1}{x} = \lim_{x\to +\infty} \frac{1}{x} = 0.$$

By the sandwich theorem, $\lim_{x\to +\infty} \frac{\sin x}{x} = 0.$

(c) $\lim_{x\to +\infty} (x - \sqrt{x^2 + 1})$

$$= \lim_{x\to +\infty} \frac{(x - \sqrt{x^2+1})(x + \sqrt{x^2+1})}{x + \sqrt{x^2+1}}$$

$$= \lim_{x\to +\infty} \frac{1}{x + \sqrt{x^2+1}}$$

$$= 0.$$

3. (15 marks) Let a_n be the sequence defined by

$$\begin{cases} a_{n+1} = 1 - (a_n - 1)^2, \text{ for } n \geq 1 \\ a_1 = \dfrac{1}{100}. \end{cases}$$

(a) Show that $0 \leq a_n \leq 1$ for any $n \geq 1$.

Solution:

We already have $0 \leq a_1 \leq 1$. Suppose that $0 \leq a_n \leq 1$, then

$-1 \leq a_n - 1 \leq 0$

$\Rightarrow 0 \leq (a_n - 1)^2 \leq 1$

$\Rightarrow -1 \leq -(a_n - 1)^2 \leq 0$

$\Rightarrow 0 \leq 1 - (a_n - 1)^2 \leq 1$

$\Rightarrow 0 \leq a_{n+1} \leq 1$.

By induction,

$0 \leq a_n \leq 1$ for any $n \geq 1$.

(b) Show that $a_{n+1} - a_n \geq 0$ for any $n \geq 1$.

Solution:

$a_{n+1} - a_n = 1 - (a_n - 1)^2 - a_n = -a_n^2 + a_n = a_n(1 - a_n)$.

Since $0 \leq a_n \leq 1$, $a_n(1 - a_n) \geq 0$. Then $a_{n+1} - a_n \geq 0$.

(c) Explain whether the limit of a_n exists and find the limit if it exists.

Solution:

Since the sequence a_n is bounded and monotonic increasing, the limit exist.

Let the limit be a_o.

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} [(1 - (a_n - 1)^2] = 1 - (\lim\limits_{n\to\infty} a_n - 1)^2$

$a_o = 1 - (a_o - 1)^2 \Rightarrow a_o^2 - a_o = 0 \Rightarrow a_o = 1$ or 0.

But the sequence is monotonic increasing and $a_1 > 0$, so $a_o = 1$.

4. (20 marks) Let n be a positive integer. Let:

$$f(x) = \begin{cases} x^3, & \text{if } x < 0; \\ x^n, & \text{if } x \geq 0. \end{cases}$$

(a) Find $f'(x)$ for $x > 0$.

Solution:

$f'(x) = nx^{n-1}$.

(b) Find all positive integers n such that:

i. $f'(0)$ exists.

ii. $f''(0)$ exists.

(Recall that by definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.)

Justify your answers.

Solution:

$n \geq 2$ for (i).

$n \geq 3$ for (ii)

When $n = 1$,

$$f(x) = \begin{cases} x^3, & \text{if } x < 0; \\ x, & \text{if } x \geq 0. \end{cases} \text{ and } f'(x) = \begin{cases} 3x^2, & \text{if } x < 0; \\ 1, & \text{if } x > 0. \end{cases}$$

$f'(0)$ does not exist at $x = 0$ since right limit does not equal left limit. So $f''(0)$ does not exist.

When $n = 2$,

$$f(x) = \begin{cases} x^3, & \text{if } x < 0; \\ x^2, & \text{if } x \geq 0. \end{cases} \text{ and } f'(x) = \begin{cases} 3x^2, & \text{if } x < 0; \\ 2x, & \text{if } x \geq 0. \end{cases}$$

In particularly, $f'(0) = 0$.

$$\lim_{h \to 0^-} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0^-} \frac{3h^2 - 0}{h} = 0$$

$$\lim_{h \to 0^+} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0^+} \frac{2h - 0}{h} = 2$$

So $f''(0)$ does not exist.

When $n \geq 3$,

$$f(x) = \begin{cases} x^3, & \text{if } x < 0; \\ x^n, & \text{if } x \geq 0. \end{cases} \text{ and } f'(x) = \begin{cases} 3x^2, & \text{if } x < 0; \\ nx^{n-1}, & \text{if } x \geq 0. \end{cases}$$

$$\lim_{h \to 0^-} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0^-} \frac{3h^2 - 0}{h} = 0$$

$$\lim_{h \to 0^+} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0^+} \frac{nh^{n-1} - 0}{h} = 0$$

So $f''(0)$ exist.

5. (15 marks) Let f be a function which is continuous on $[a, b]$, differentiable in (a, b) and satisfies $f(a) = f(b) = 0$. By considering the function $e^{x/s} f(x)$, show that for any non-zero real number s there exists $d \in (a, b)$ satisfying

$$sf'(d) + f(d) = 0.$$

Solution:
Let $h(x) = e^{x/s} f(x)$, then $h(x)$ is a function which is continuous on $[a, b]$, differentiable in (a, b) too and satisfies $h(a) = h(b) = 0$.
By the mean-value theorem, there exists $d \in (a, b)$ satisfying
$h'(d) = \dfrac{h(b) - h(a)}{b - a} = 0$
$h'(d) = (1/s)e^{d/s} f(d) + e^{d/s} f'(d) = 0$.
Since $e^{d/s} \neq 0$, $sf'(d) + f(d) = 0$.

6. (15 marks) Determine whether there is any function satisfying all of the following conditions:

   (i) f is differentiable in $(0, 2)$.

   (ii) f is continuous on $[0, 2]$.

   (iii) f satisfies $f(0) = 1, f(2) = 10, f'(x) \leq 2,$ for each $x \in (0, 2)$.

   Give an example of such a function if you think "yes", or explain why no such functions exist if you think "no".

   Solution:

   No such functions exist.

   Suppose there is such a function f satisfying conditions (i) and (ii).

   By the mean-value theorem, there exist $c \in (0, 2)$ satisfying

   $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{10 - 1}{2 - 0} = \dfrac{9}{2} > 2$, which does not satisfy (iii).