THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2016-2017
Midterm Examination

Name (in print): ____

Student ID: _______________ Programme: ___________ Section: MATH1010____

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INSTRUCTIONS to students:

1. Answer all questions. Show work to justify all answers.
2. The examination lasts 90 minutes.
3. There are a total of 80 points.
4. Answer the questions in the space provided.

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FOR MARKERS' USE ONLY:

1	
2	
3	
4	
5	
6	
Total	/80 points

1. (12 marks) Evaluate the following.

(a) $\lim_{x\to 1} \dfrac{3-x-x^2-x^3}{1-x} =$

Solution:

$\lim_{x\to 1} \dfrac{3-x-x^2-x^3}{1-x}$ ($\frac{0}{0}$ type)

$= \lim_{x\to 1} \dfrac{-1-2x-3x^2}{-1}$ (By L'Hopital Rule)

$= 6$

(b) $\lim_{x\to +\infty} \dfrac{e^{2x}+x^3\cos x}{e^{2x}-x^3\sin x} =$

Solution:

$\dfrac{e^{2x}+x^3\cos x}{e^{2x}-x^3\sin x}$

$= \dfrac{1+\frac{x^3\cos x}{e^{2x}}}{1-\frac{x^3\sin x}{e^{2x}}}$

$\to 1$ as $x \to +\infty$

Since $\sin x$ and $\cos x$ are bounded functions and $\forall k \in \mathbb{N}, \frac{x^k}{e^x} \to 0$ as $x \to \infty$

(c) $\lim_{x\to +\infty} (x-\sqrt{x^2-8x+3}) =$

Solution:

$x-\sqrt{x^2-8x+3} = \dfrac{x^2-x^2+8x-3}{x+\sqrt{x^2-8x+3}} = \dfrac{8-\frac{3}{x}}{1+\sqrt{1-\frac{8}{x}+\frac{3}{x^2}}} \to 4$ as $x \to +\infty$

2. (16 marks) Find $\dfrac{dy}{dx}$ if

(a) $y = \dfrac{e^{2x}}{1+x}$

Solution:
$$\frac{dy}{dx} = \frac{(1+x)(2e^{2x}) - e^{2x}(1)}{(1+x)^2} = \frac{e^{2x}(1+2x)}{(1+x)^2}$$

(b) $y = \ln(2 + \sin(1+3x))$

Solution:
$$\frac{dy}{dx} = \frac{1}{2+\sin(1+3x)} \cos(1+3x)(3) = \frac{3\cos(1+3x)}{2+\sin(1+3x)}$$

(c) $xy^2 + \cos(x+y) = 1$

Solution:
$$y^2 + x(2y)\frac{dy}{dx} + \sin(x+y)(1 + \frac{dy}{dx}) = 0$$
$$(2xy + \sin(x+y))\frac{dy}{dx} = -y^2 - \sin(x+y)$$
$$\frac{dy}{dx} = \frac{-y^2 - \sin(x+y)}{2xy + \sin(x+y)}$$

(d) $y = (\ln x)^x$

Solution:
$$\frac{dy}{dx} = \frac{d}{dx} e^{x\ln(\ln x)} = e^{x\ln(\ln x)} \left(\ln(\ln x) + \frac{x}{\ln x} \cdot \frac{1}{x} \right) = (\ln x)^{x-1}(\ln x \ln(\ln x) + 1)$$

3. (10 marks) Evaluate the following limits.

(a) $\lim_{x\to 0} \dfrac{\tan^{-1} x}{1-\sqrt{1-x}}$ ($\tan^{-1} x = \arctan x$ is the inverse of tangent.)

Solution:

$\lim_{x\to 0} \dfrac{\tan^{-1} x}{1-\sqrt{1-x}}$ $(\frac{0}{0}$ type)

$= \lim_{x\to 0} \dfrac{\frac{1}{1+x^2}}{-\frac{1}{2}\frac{-1}{\sqrt{1-x}}}$ (By L'Hopital Rule)

$= 2$

(b) $\lim_{x\to 0} \left(\dfrac{1}{\ln(1+x)} - \dfrac{1}{\sin x} \right)$

Solution:

$\lim_{x\to 0} \left(\dfrac{1}{\ln(1+x)} - \dfrac{1}{\sin x} \right)$

$= \lim_{x\to 0} \dfrac{\sin x - \ln(1+x)}{\sin x \ln(1+x)}$ $(\frac{0}{0}$type)

$= \lim_{x\to 0} \dfrac{\cos x - \frac{1}{1+x}}{\cos x \ln(1+x) + \frac{\sin x}{1+x}}$ (By L'Hopital Rule)

$= \lim_{x\to 0} \dfrac{(x+1)\cos x - 1}{(x+1)\cos x \ln(x+1) + \sin x}$ $(\frac{0}{0}$type)

$= \lim_{x\to 0} \dfrac{\cos x - (1+x)\sin x}{\cos x \ln(1+x) - (1+x)\sin x \ln(1+x) + (1+x)\cos x \frac{1}{1+x} + \cos x}$

(By L'Hopital Rule)

$= \dfrac{1}{2}$

4. (12 marks) Let a_n be the sequence defined by

$$\begin{cases} a_{n+1} = 3 - \dfrac{1}{a_n}, \text{ for } n \geq 1 \\ a_1 = 1. \end{cases}$$

(a) Show that $1 \leq a_n \leq 3$ for any $n \geq 1$.

(b) Show that $a_{n+1} - a_n > 0$ for any $n \geq 1$.

(c) Explain whether the limit of a_n exists and find the limit if it exists.

Solution:

(a) $\forall n \geq 1$, let P(n)be the proposition "$1 \leq a_n \leq 3$".

$a_1 = 1$

Hence, P(1)is true.

Assume $\exists k \geq 1$ s.t. P(k) is true. i.e. $1 \leq a_k \leq 3$

Then, when $n = k+1$,

$1 \leq 3 - \dfrac{1}{1} \leq 3 - \dfrac{1}{a_k} = a_{k+1} \leq 3 - \dfrac{1}{3} \leq 3$ (By assumption)

Hence, P($k+1$) is also true.

By the principle of Mathematical Induction, P(n) is true $\forall n \geq 1$.

(b) $\forall n \geq 1$, let P(n)be the proposition "$a_{n+1} - a_n > 0$".

When $n = 1$,

$a_2 = 3 - \frac{1}{1} = 2$

$a_2 - a_1 = 1$

Hence, P(1)is true.

Assume $\exists k \geq 1$ s.t. P(k) is true. i.e. $a_{k+1} - a_k > 0$

Then, when $n = k+1$,

$a_{k+2} - a_{k+1} = 3 - \dfrac{1}{a_{k+1}} - 3 + \dfrac{1}{a_k} = \dfrac{a_{k+1} - a_k}{a_k a_{k+1}} > 0$ (By assumption and (a))

Hence, P($k+1$) is also true.

By the principle of Mathematical Induction, P(n) is true $\forall n \geq 1$.

(c) By (a), a_n is bounded, by (b), a_n is monotonically increasing.

By Monotone Convergence Theorem, limit of a_n exists.

Let L denote limit of a_n,

By (a), $1 \leq L \leq 3$

$L = 3 - \dfrac{1}{L}$

$L^2 - 3L + 1 = 0$

$L = \dfrac{3 - \sqrt{5}}{2}$(rejected) or $L = \dfrac{3 + \sqrt{5}}{2}$

5. (15 marks) Let

$$f(x) = \begin{cases} x^2 \sin(\ln|x|), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Write down the first derivative of the function $\ln|x|$ for $x \neq 0$. No working steps or proof is required.

(b) Find $f'(x)$ for $x \neq 0$.

(c) Find $f'(0)$.

(d) Explain whether $f'(x)$ is differentiable at $x = 0$.

Solution:

(a) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(b) $\forall x \neq 0,\ f'(x) = 2x\sin(\ln|x|) + x^2\cos(\ln|x|)\frac{1}{x} = x(2\sin(\ln|x|) + \cos(\ln|x|))$

(c) Consider $\frac{f(h) - f(0)}{h} = \frac{h^2\sin(\ln|h|)}{h} = h\sin(\ln|h|) \to 0$ as $h \to 0$
since $\sin x$ is a bounded function.
i.e. $f'(0) = 0$

(d) Consider $\frac{f'(h) - f'(0)}{h}$
$= \frac{h(2\sin(\ln|h|) + \cos(\ln|h|)) - 0}{h}$ (By (b) and(c))
$= 2\sin(\ln|h|) + \cos(\ln|h|)$
$\forall n \geq 1$, let $x_n = \exp(-n\pi)$.
$x_n \to 0$ as $n \to \infty$.
Then, consider $\frac{f'(x_n) - f'(0)}{x_n}$
$= 2\sin(\ln|x_n|) + \cos(\ln|x_n|)$
$= 2\sin(-n\pi) + \cos(-n\pi)$
$= (-1)^n$
$(-1)^n$ is not a convergent sequence.
Therefore, $f'(x)$ is NOT differentiable at $x = 0$.

6 (15 marks) Let $f(x)$ be a function such that $f'(x)$ is strictly decreasing.

(a) Prove that $f'(x+1) < f(x+1) - f(x) < f'(x)$ for any x.

(b) Prove that

$$f'(1) + f'(2) + f'(3) < f(3) - f(0) < f'(0) + f'(1) + f'(2).$$

Solution:

(a) By Mean Value Theorem,
$\exists c \in (x, x+1)$ s.t.
$f'(c) = \dfrac{f(x+1) - f(x)}{x+1-x} = f(x+1) - f(x)$
Since $f'(x)$ is strictly increasing,
$f'(x+1) < f'(c) = f(x+1) - f(x) < f'(x)$

(b) By (a), we have
$f'(3) < f(3) - f(2) < f'(2)$
$f'(2) < f(2) - f(1) < f'(1)$
$f'(1) < f(1) - f(0) < f'(0)$
Sum these 3 inequalities up, we have,

$$f'(1) + f'(2) + f'(3) < f(3) - f(0) < f'(0) + f'(1) + f'(2).$$

END OF PAPER