THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010A-H University Mathematics 2015-2016
Midterm Examination

Name (in print): ____

Student ID: ________________ Programme: ________________ Section: MATH1010____

* * *

INSTRUCTIONS to students:

1. Answer all questions. Show work to justify all answers.
2. The examination lasts 90 minutes.
3. There are a total of 100 points.
4. Answer the questions in the space provided.

* * *

FOR MARKERS' USE ONLY:

1		6	
2		7	
3		8	
4		9	
5			
		Total	/100 points

1. Find $\frac{dy}{dx}$ for the following functions without starting from first principle.

   (a) $y = e^{x^2+1}$

   (b) $y = \frac{e^{x^2+1}}{x}$

(8 points)

2. Let $f(x) = \dfrac{|x-1|}{x}$. Find

(a) $\lim_{x \to 2} f(x)$;

(b) $\lim_{x \to -\infty} f(x)$.

(6 points)

3. Let C be a curve given by the function $f(x) = xe^x$. Find the equation of the tangent of C at $x = 1$.

(6 points)

4. Evaluate the following limits.

(a) $\lim_{x\to-\infty}(x+\sqrt{x^2+6x+2})$

(b) $\lim_{x\to 0} x^3 \sin\left(\frac{1}{e^x - e^{-x}}\right)$

(10 points)

5. Let $f(x)$ be a differentiable function such that $f(x) \neq 0$ for all real numbers x. Use the definition of derivative (first principle) to prove that

$$\frac{d}{dx}\left(\frac{1}{(f(x))^2}\right) = -\frac{2f'(x)}{(f(x))^3}.$$

(You may use, without proof, the result that if a function is differentiable at a certain point then that function is continuous at the same point.)

(10 points)

6. Find $\dfrac{dy}{dx}$ if

$$xy + \ln(x^2 + y^2 + 100) = 1.$$

(5 points)

7. Let $f(x) = |x| \sin^2 x$.

(a) Find $f'(x)$ for $x \neq 0$.

(b) Show that the function $f(x)$ is differentiable at $x = 0$, and find $f'(0)$.

(c) Explain whether $f'(x)$ is continuous at $x = 0$.

(d) Explain whether $f'(x)$ is differentiable at $x = 0$.

(20 points)

9

8. (a) Suppose that $0 < a < b$. Prove that

$$(1 + \ln a)(b - a) < b \ln b - a \ln a < (1 + \ln b)(b - a).$$

(b) Suppose that $0 < a < b$. Prove that there exists $c \in (a, b)$ such that

$$ae^b - be^a = (a - b)(1 - c)e^c.$$

(15 points)

9. Let $a > 1$ and p, q, r, s be four real numbers such that

$$p + q = r + s \qquad \text{and} \qquad 0 < p - q < r - s.$$

(a) Show that the function $f(x) = a^x - a^{-x}$ is strictly increasing for $x > 0$.

(b) By considering the value of $f(x)$ at $\frac{1}{2}(p-q)$ and $\frac{1}{2}(r-s)$, prove that

$$a^p - a^q < a^r - a^s.$$

Hence, deduce that if $u > v > 0$, then

$$u^p v^q - u^q v^p < u^r v^s - u^s v^r.$$

(20 points)

13

END OF PAPER