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**Department of Mathematics**  
**Exercise for MATH1010 University Mathematics**  
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# 1 Preliminaries

## 1.1 Functions

1. Graph the functions  $f(x) = \frac{x}{2}$  and  $g(x) = \frac{4}{x} - 1$  together, to identify values of  $x$  for which

$$\frac{x}{2} > \frac{4}{x} - 1.$$

Confirm your answer by solving the inequality algebraically.

2. Plot all points  $(x, y)$  on the plane that satisfy  $x^2 + y^2 + 2x - 4y - 14 = 0$ . Explain why it is not the graph of any function.
3. Find all real numbers  $x$  satisfying

$$2^x + 2^{1-x} = 3.$$

4. Determine whether each of the following statement is true or false.

- (a)  $e^{x+y} = e^x + e^y$  for all real numbers  $x$  and  $y$
- (b)  $\ln(x+y) = (\ln x)(\ln y)$  for all  $x, y > 0$
- (c)  $e^{xy} = e^x + e^y$  for all real numbers  $x$  and  $y$
- (d)  $\ln(x^y) = (\ln x)^y$  for all  $x, y > 0$
- (e) If  $h(x) = f(x)g(x)$ , then  $h'(x) = f'(x)g'(x)$ .
- (f)  $\frac{d}{dx}(2^x) = x2^{x-1}$
- (g)  $\int \ln x \, dx = \frac{1}{x} + C$

## 1.2 Limits and Derivatives

1. Evaluate the following limits

- (a)  $\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{4x - x^2}$
- (b)  $\lim_{x \rightarrow 0} \frac{x^{928}}{\sqrt{x^{928} + 4} - 2}$
- (c)  $\lim_{t \rightarrow 0} \left( \frac{1}{2t} - \frac{1}{t^2 + 2t} \right)$
- (d)  $\lim_{t \rightarrow 0} \frac{\ln(e+t) - 1}{et}$
- (e)  $\lim_{x \rightarrow -2} \frac{|x|^3 - 8}{x^4 - 16}$

2. Use first principles to find  $\frac{dy}{dx}$  of the following functions.

- (a)  $y = x^3 - 4x$
- (b)  $y = 3 - 2\sqrt{x}$
- (c)  $y = \frac{6}{x^2}$
- (d)  $y = \frac{1}{\sqrt{x}}$

3. Find the first derivative of the following functions.

(a)  $f(x) = \ln(x^\pi + e^\pi)$

(d)  $u(x) = \frac{x}{\sqrt{e^{-x} + \sqrt{x}}}$

(b)  $g(x) = \frac{1}{\sqrt{\pi^x + x^4}}$

(e)  $v(x) = \ln \frac{(1+x^2)^{87}}{x^2}$

(c)  $h(x) = xe^{1/x^2} + \frac{\ln(2x)}{\sqrt{x}}$

4. If  $t = (x+1)(x+2)^2(x+3)^3$  and  $\ln t = e^y$ , find  $\frac{dy}{dx}$  in terms of  $x$  only.

5. Suppose  $f'(2) = 13$ ,  $g(7) = 2$  and  $g'(7) = 53$ . If  $y = f(g(x))$ , find  $\left. \frac{dy}{dx} \right|_{x=7}$ .

6. Suppose  $y = f(x)$  is a smooth function. Determine whether each of the following statements is true or false.

(a) If  $f$  is increasing, then  $f''(x) > 0$  for all  $x$ .

(b) If  $f'(x) > 0$  for all  $x$ , then  $f(x) > 0$  for all  $x$ .

(c) If  $f'(x) > 0$  for all  $x$ , then  $f'$  is increasing.

(d) If  $f(x) > 0$  for all  $x$ , then  $f'(x) > 0$  for all  $x$ .

(e) If  $f'(3) = 0$ , then  $f$  must have a maximum or a minimum at  $x = 3$ .

### 1.3 Applications of Derivatives

1. If  $L$  is a tangent line to the graph  $y = x^2 + 3$ , and the  $x$ -intercept of  $L$  is 1, find all possible points at which  $L$  touches the graph of  $y = x^2 + 3$ .

2. The position  $x$  of a particle at time  $t$  is given by

$$x = t^3 + at^2 + bt + c$$

for some constants  $a, b, c$ . It is known that when  $t = 0$ , the particle is at position  $x = 0$ ; also, there is a certain time  $t_0$  such that the velocity and acceleration of the particle are both zero at time  $t_0$ , and at time  $t_0$  the particle is at position  $x = 1$ . Find the values of  $a, b$  and  $c$ .

3. Water is being drained, at a constant rate of  $1/4$  cubic meter per hour, from the bottom of a container that takes the shape of an inverted regular cone. The radius of the base of the cone is 3 meters, and the height of the cone is 2 meters. What is the rate of change of the depth of the water, when the water is 1 meter deep?

4. Let  $C$  be the curve  $y = \frac{1}{x} + x$ . Note that  $P = (1, 2)$  and  $Q = (0.5, 2.5)$  are two points on  $C$ .

(a) Find equations of the tangent and normal to  $C$  at the point  $P$ .

(b) Show that the tangent to  $C$  at the point  $Q$  passes through the point  $A = (0, 4)$ .

5. A water container is made in the shape of a right circular cone, with its axis perpendicular to the ground and its apex at the bottom of the container. The axis subtends an angle of  $30^\circ$  with the (conical) surface of the container. Water is flowing out of the cone through the apex at a constant rate of  $\pi \text{ cm}^3$  per second.
- Suppose that when the depth of water is  $h$  cm, the volume of water in the container is  $V \text{ cm}^3$ . Express  $V$  in terms of  $h$ .
  - How fast is the water level falling when the depth of water is 4 cm?
6. Let  $C$  be the curve given by the equation  $y = \frac{x^2}{1+x} - \frac{4}{3}$ .
- Find the  $x$ - and  $y$ -intercepts of the curve  $C$ .
  - Find  $\frac{dy}{dx}$  and show that  $\frac{d^2y}{dx^2} = \frac{a}{(1+x)^b}$ , where  $a, b$  are integers whose respective values you have to determine.
  - Find the turning point(s) of the curve  $C$ .  
For each turning point, determine whether it is a (relative) maximum or a (relative) minimum point.
  - What are the asymptotes of the curve  $C$ , if any?
  - Sketch the curve  $C$  for  $-1 < x < 3$ .
7. The function  $f(x) = \frac{x^2 + kx + 9}{x^2 + 1}$ , where  $k$  is a constant, attains a stationary value at  $x = 3$ .
- Consider the curve  $C$  given by  $y = f(x)$ .
- Determine the value of  $k$ .
  - Find the  $x$ - and  $y$ -intercepts of the curve  $y = f(x)$ .
    - Find the (relative) maximum and (relative) minimum points of the curve  $C$ .
    - What are the asymptotes of the curve  $C$ , if any?
  - Sketch the curve  $C$  for  $-6 \leq x \leq 6$ .
8. Let  $\Sigma$  be a sphere with (fixed) radius  $R$ .
- Suppose  $\Gamma$  be a right circular cylinder inscribed in the sphere  $\Sigma$ , with height  $h$ , radius  $r$ , volume  $V$  and surface area  $A$ .
    - Express  $V$  in terms of  $r$  and  $R$  alone.
    - Express  $A$  in terms of  $r$  and  $R$  alone.
  - What is the largest possible value of  $V$ ?
  - What is the largest possible value of  $A$ ?
9. Let  $\beta \in (1, +\infty)$ . Let  $f : (0, +\infty) \rightarrow \mathbb{R}$  be the function defined by  $f(x) = x^\beta + \beta - 1 - \beta x$  for any  $x \in (0, +\infty)$ .

- (a) i. Compute  $f'$ .  
 ii. Show that  $f$  is strictly decreasing on  $(0, 1]$ .  
 iii. Show that  $f$  is strictly increasing on  $[1, +\infty)$ .  
 iv. Determine whether  $f$  attains the maximum and/or the minimum on  $(0, +\infty)$ .  
 (b) Hence, or otherwise, show that  $(1+r)^\beta \geq 1 + \beta r$  for any  $r \in (-1, +\infty)$ .

## 1.4 Integration

1. Evaluate the following indefinite integrals.

$$(a) \int (3 - x^2)^3 dx \quad (c) \int \frac{x+1}{\sqrt{x}} dx \quad (d) \int \left(8t - \frac{2}{t^{\frac{1}{4}}}\right) dt$$

$$(b) \int x^2(5 - x)^4 dx$$

2. Use a suitable substitution to evaluate the following integral.

$$(a) \int \frac{dx}{\sqrt{2-5x}} \quad (d) \int x^2 \sqrt[3]{1+x^3} dx \quad (g) \int \frac{e^x dx}{2+e^x}$$

$$(b) \int \frac{e^{3x} + 1}{e^x + 1} dx \quad (e) \int \frac{xdx}{(1+x^2)^2} \quad (h) \int \frac{dx}{1+e^x}$$

$$(c) \int \frac{x}{\sqrt{1-x^2}} dx \quad (f) \int xe^{-x^2} dx$$

3. Evaluate the following definite integrals.

$$(a) \int_1^3 \frac{2x^3 - 5}{x^2} dx \quad (c) \int_0^1 \frac{5x}{(4+x^2)^2} dx \quad (d) \int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$$

$$(b) \int_0^1 x\sqrt{1-x^2} dx$$

4. Find the area of the regions bounded by the graphs of the given functions.

$$(a) y = 4 - x^2; x\text{-axis} \quad (d) y = x^2; y = x - 2; x\text{-axis}$$

$$(b) y = 3 - x^2; y = -x - 3 \quad (e) y = \sqrt{x}; y = x - 2; x\text{-axis}$$

$$(c) y = x^2 - 4; y = -x^2 - 2x \quad (f) x + y^2 = 4; x + y = 2$$

5. The slope at any point  $(x, y)$  of a curve  $C$  is given by  $\frac{dy}{dx} = 4 - 2x$  and  $C$  passes through the point  $(1, 0)$ .

- (a) Find an equation of the curve  $C$ .  
 (b) Find the area of the finite region bounded by the curve  $C$  and the  $x$ -axis.

6. The slope of the tangent to a curve  $C$  at any point  $(x, y)$  on  $C$  is  $x^2 - 2$ .  $C$  passes through the point  $(3, 4)$ .

- (a) Find an equation of the curve  $C$ .
- (b) Find the coordinates of the point on  $C$  at which the slope of the tangent is  $-2$ .
7. The curve  $y = x^3 - x^2 - 2x$  cuts the  $x$ -axis at the origin and the points  $(a, 0)$ ,  $(b, 0)$ , where  $a < 0 < b$ .
- (a) Find the values of  $a$  and  $b$ .
- (b) Find the total area of the region bounded by the curve and the  $x$ -axis.
8. The slope at any point  $(x, y)$  of a curve  $C$  is given by  $\frac{dy}{dx} = 6x + \frac{1}{x^2}$ , where  $x > 0$ . Suppose the curve  $C$  cuts the  $x$ -axis at the point  $(1, 0)$ . Find its equation.

## 2 Differentiation

### 2.1 Graphs of Functions

1. Sketch the graphs of the following functions.

(a)  $f(x) = x^2 - 4x - 5$

(b)  $f(x) = x^3 - 4x$

(c)  $f(x) = x^3 + x^2 - 5x + 3$

(d)  $f(x) = x + \frac{1}{x}$

(e)  $f(x) = \frac{2x - 9}{x + 3}, x \neq -3$

(f)  $f(x) = \frac{x^2}{x - 2}, x \neq 2$

(g)  $f(x) = 3 - \sqrt{4 - x^2}, -2 \leq x \leq 2$

(h)  $f(x) = \frac{x}{\sqrt{x^2 + 1}}$

(i)  $f(x) = 3 - e^x$

(j)  $f(x) = \frac{e^x - e^{-x}}{2}$

(k)  $f(x) = 2 + \ln(x + 1), x > -1$

(l)  $f(x) = 5 - \ln(x - 2)^2, x \neq 2$

(m)  $f(x) = 1 - 2 \sin x$

(n)  $f(x) = 2 + \cos 2x$

(o)  $f(x) = 3 \cos \left(x + \frac{\pi}{3}\right)$

(p)  $f(x) = \cos x - \sin x$

(q)  $f(x) = 1 + \sin^2 x$

(r)  $f(x) = 1 + \cos x \sin x$

(s)  $f(x) = x \cos x$

2. Sketch the graphs of the following functions.

(a)  $f(x) = |x^2 - 2x - 3|$

(b)  $f(x) = x^2 - 4|x| + 3$

(c)  $f(x) = ||x - 2| - 4|$

(d)  $f(x) = |3 - |x^2 - 1||$

(e)  $f(x) = |2 + 3 \sin x|$

(f)  $f(x) = 2 - |\sin x|$

3. Sketch the graphs of the following piecewise defined functions.

(a)  $f(x) = \begin{cases} 2x + 5, & x < -1 \\ x^2 - 1, & x \geq -1 \end{cases}$

(c)  $f(x) = \begin{cases} 1 - x^2, & x < 1 \\ \ln x, & x \geq 1 \end{cases}$

(b)  $f(x) = \begin{cases} x, & x < -2 \\ -x, & -2 \leq x < 2 \\ x, & x \geq 2 \end{cases}$

(d)  $f(x) = \begin{cases} 1 - |x + 3|, & x < -2 \\ 2 - |x|, & -2 \leq x < 2 \\ 1 - |x - 3|, & x \geq 2 \end{cases}$

### 2.2 Limits and Continuity

1. Evaluate the following limits

(a)  $\lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}$

(c)  $\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos x}$

(b)  $\lim_{x \rightarrow 0} \frac{\sin 3x}{4x}$

(d)  $\lim_{x \rightarrow 0} \frac{\sin^2 x}{1 - \cos x}$

$$\begin{array}{ll}
 \text{(e)} \lim_{x \rightarrow 0} \frac{\sin 9x - \sin 3x}{\sin 2x} & \text{(i)} \lim_{x \rightarrow \frac{\pi}{4}} \tan 2x \tan \left( \frac{\pi}{4} - x \right) \\
 \text{(f)} \lim_{x \rightarrow 0} \frac{\cos x - \cos 3x}{x^2} & \text{(j)} \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{1 - 2 \cos x} \\
 \text{(g)} \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x} & \text{(k)} \lim_{t \rightarrow 0} \frac{\ln(1+t)}{t} \\
 \text{(h)} \lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3} & \text{(l)} \lim_{h \rightarrow 0} \frac{e^h - 1}{h}
 \end{array}$$

2. Evaluate the following limits.

$$\begin{array}{ll}
 \text{(a)} \lim_{x \rightarrow \infty} \frac{2x + 5}{x^2 + 3} & \text{(k)} \lim_{x \rightarrow +\infty} \frac{8e^{-9x} + x^2}{6e^{4x} - x^2} \\
 \text{(b)} \lim_{x \rightarrow \infty} \frac{3x + 1}{5x - 4} & \text{(l)} \lim_{x \rightarrow +\infty} \frac{e^x + x^2}{e^x - x^2} \\
 \text{(c)} \lim_{x \rightarrow \infty} \frac{6x^2 + 2x - 5}{2x^2 - 4x + 1} & \text{(m)} \lim_{x \rightarrow +\infty} \frac{\ln(x^2 + x - 1)}{\ln(x^8 - x + 1)} \\
 \text{(d)} \lim_{x \rightarrow \infty} \frac{2x^3 - 5x + 3}{3x^3 - x^2 + 6} & \text{(n)} \lim_{x \rightarrow +\infty} \frac{\ln(x^2 + e^{3x})}{\ln(x^3 + e^x)} \\
 \text{(e)} \lim_{x \rightarrow +\infty} \frac{\sqrt{9x^4 - 3x + 2}}{x^2 - 2x + 5} & \text{(o)} \lim_{x \rightarrow +\infty} \frac{\ln(1 + 8^x)}{\ln(1 + 2^x)} \\
 \text{(f)} \lim_{x \rightarrow +\infty} (\sqrt{4x^2 + 5x} - 2x) & \text{(p)} \lim_{x \rightarrow -\infty} \frac{\ln(1 + 8^x)}{\ln(1 + 2^x)} \\
 \text{(g)} \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 4x}) & \text{(q)} \lim_{x \rightarrow \infty} \frac{\sin x}{x} \\
 \text{(h)} \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1}) & \text{(r)} \lim_{x \rightarrow +\infty} \frac{\cos x}{(\ln x)^2} \\
 \text{(i)} \lim_{x \rightarrow +\infty} \frac{\sqrt{9x^4 - 3x + 2}}{x^2 - 2x + 5} & \\
 \text{(j)} \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 4x}) &
 \end{array}$$

## 2.3 Derivatives

1. For each of the following functions, determine whether it is differentiable at  $x = 0$ . Find  $f'(0)$  if it is.

$$\begin{array}{lll}
 \text{(a)} f(x) = x^{\frac{4}{3}} & \text{(b)} f(x) = |\sin x| & \text{(c)} f(x) = x|x| \\
 \text{(d)} f(x) = \begin{cases} 5 - 2x, & \text{when } x < 0 \\ x^2 - 2x + 5, & \text{when } x \geq 0 \end{cases} & & \\
 \text{(e)} f(x) = \begin{cases} 1 + 3x - x^2, & \text{when } x < 0 \\ x^2 + 3x + 2, & \text{when } x \geq 0 \end{cases} & & \\
 \text{(f)} f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases} & &
 \end{array}$$

2. Find all values of  $a$  and  $b$  for which the function

$$f(x) = \begin{cases} \sin x, & \text{for } x < \pi \\ ax + b, & \text{for } x \geq \pi \end{cases}$$

is differentiable at  $x = \pi$ .

3. Let

$$f(x) = \begin{cases} \frac{1 - \cos x}{x^2}, & \text{for } x < 0 \\ ax + b, & \text{for } x \geq 0 \end{cases}$$

Find the value of  $a$  and  $b$  if  $f(x)$  is differentiable at  $x = 0$ .

4. Find the first derivatives of the following functions.

- |   |                                    |                                     |
|---|------------------------------------|-------------------------------------|
| (a) $y = x^3 - 4x + 3$                  | (j) $y = \frac{\sin x}{x}$         | (s) $y = \ln(x + \sqrt{1 + x^2})$   |
| (b) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ | (k) $y = \frac{\tan x}{\sqrt{x}}$  | (t) $y = \sqrt{x + \sqrt{x}}$       |
| (c) $y = x^2 e^{5x}$                    | (l) $y = (x^2 + 1)^7$              | (u) $y = \cosh^2 x$                 |
| (d) $y = \cos x \ln x$                  | (m) $y = \sqrt{x^4 + 1}$           | (v) $y = \frac{\sinh^2 x}{\cosh x}$ |
| (e) $y = \sin x \cos x$                 | (n) $y = \cos(x^2)$                | (w) $y = \ln(\sinh x)$              |
| (f) $y = 3 \sec x - \tan x$             | (o) $y = x e^{x^3 + x}$            | (x) $y = \sin^{-1} \sqrt{x}$        |
| (g) $y = x \cot x$                      | (p) $y = \ln(\ln x)$               | (y) $y = \cos(\tan^{-1} x)$         |
| (h) $y = \frac{3x - 4}{x + 2}$          | (q) $y = e^{\sin x}$               |                                     |
| (i) $y = \frac{x^2 + 1}{x + 1}$         | (r) $y = \frac{x}{\sqrt{1 + x^2}}$ |                                     |

5. Find the first derivatives of the following functions.

- |                      |                        |                             |
|----------------------|------------------------|-----------------------------|
| (a) $y = 3^x$        | (c) $y = x^x$          | (e) $y = (\cos x)^{\sin x}$ |
| (b) $y = 2^{\cos x}$ | (d) $y = x^{\sqrt{x}}$ | (f) $y = x^{x^x}$           |

6. Find  $\frac{dy}{dx}$  for the following implicit functions.

- |                        |                       |   |
|------------------------|-----------------------|---|
| (a) $x^2 + y^2 = 4$    | (c) $x^3 + y^3 = 2xy$ | (e) $\sin(xy) = (x + y)^2$                      |
| (b) $x^3 y + xy^2 = 1$ | (d) $x e^{xy} = 1$    | (f) $\cos\left(\frac{y}{x}\right) = \ln(x + y)$ |

7. Find  $\frac{d^2 y}{dx^2}$  for the following functions.

- |                                    |                       |                            |
|------------------------------------|-----------------------|----------------------------|
| (a) $y = \sqrt{x} e^{x^2}$         | (d) $y = \sec x$      | (g) $x^4 y - 3x^2 y^3 = 5$ |
| (b) $y = \frac{x}{\sqrt{1 + x^2}}$ | (e) $y = \tan^{-1} x$ | (h) $y = 3^{x^2}$          |
| (c) $y = (\ln x)^2$                | (f) $x^2 + y^3 = 1$   | (i) $y = x^{\ln x}$        |

8. Prove that the Chebyshev polynomials

$$T_m(x) = \frac{1}{2^{m-1}} \cos(m \cos^{-1} x), \quad m = 0, 1, 2, \dots$$

satisfy

$$(1 - x^2)T_m''(x) - xT_m'(x) + m^2T_m(x) = 0$$

9. Prove that the Legendre polynomials

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m \quad m = 0, 1, 2, \dots$$

satisfy

$$(1 - x^2)P_m''(x) - 2xP_m'(x) + m(m + 1)P_m(x) = 0$$

10. Show that the function

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is not differentiable at  $x = 0$ .

11. Show that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is differentiable at  $x = 0$  but  $f'(x)$  is not continuous at  $x = 0$ .

## 2.4 Mean Value Theorem and Taylor's Theorem

1. Using the mean value theorem to prove for  $0 < y < x$  and  $p > 1$ ,

$$py^{p-1}(x - y) < x^p - y^p < px^{p-1}(x - y).$$

2. Using the mean value theorem to prove that for  $0 \leq x_1 < x_2 < x_3 \leq \pi$ ,

$$\frac{\sin x_2 - \sin x_1}{x_2 - x_1} > \frac{\sin x_3 - \sin x_2}{x_3 - x_2}.$$

3. Using the mean value theorem to prove that for  $x > 0$ ,

$$\frac{x}{1+x} < \ln(1+x) < x.$$

Hence, deduce that for  $x > 0$ ,

$$\frac{1}{1+x} < \ln \left( 1 + \frac{1}{x} \right) < \frac{1}{x}.$$

4. By applying the mean value theorem, prove that the equation

$$a_1x + a_2x^2 + \cdots + a_nx^n = \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}$$

has a root between 0 and 1.

5. Let  $f(x)$  be a function defined on  $[0, \infty)$  such that

- $f(0) = 0$ ,
- $f'(x)$  exists and is monotonic increasing on  $(0, \infty)$ .

Prove that

$$f(a+b) \leq f(a) + f(b)$$

for  $0 \leq a \leq b \leq a+b$ .

6. Let  $A$  be a subset of  $\mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$  be a function. Suppose that there exists  $L > 0$  such that

$$|f(x) - f(y)| \leq L|x - y|$$

for any  $x, y \in A$ , then  $f$  is said to be satisfying the **Lipschitz condition** on  $A$ .

Prove that  $\sin x$  satisfies the Lipschitz condition on  $\mathbb{R}$ .

7. Let  $f(x)$  be a function such that  $f'(x)$  is strictly decreasing for  $x > 0$ .

- (a) Using the mean value theorem, show that

$$f'(k+1) < f(k+1) - f(k) < f'(k) \text{ and } k \geq 1.$$

- (b) Using (a), prove that for any integer  $n \geq 2$ ,

$$f'(2) + f'(3) + \cdots + f'(n) < f(n) - f(1) < f'(1) + f'(2) + \cdots + f'(n-1).$$

8. Find the Taylor polynomials of the given orders for the following functions at  $x = 0$ .

- (a)  $\ln(1+x)$ , order = 4  
 (b)  $\cosh x = \frac{e^x + e^{-x}}{2}$ , order = 6  
 (c)  $\sinh x = \frac{e^x - e^{-x}}{2}$ , order = 5

9. Find the Taylor Series for the following functions at the given points.

- (a)  $\frac{1}{1+x}$  at  $x = 0$   
 (b)  $\sin x$  at  $x = \frac{\pi}{2}$ .  
 (c)  $e^x$  at  $x = 1$ .

10. (a) Write down the Taylor polynomial  $P_3(x)$  of degree 3 generated by  $f(x) = \ln(1-x)$  at 0  
 (b) Hence approximate  $\ln 0.99$ .

11. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function that can be differentiated infinitely many times and the Taylor series expansion of  $f$  at  $x = 0$  is

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$

- (a) If  $f$  is an odd function, show that  $a_0 = a_2 = a_4 = \cdots = 0$ .  
 (b) If  $f$  is an even function, show that  $a_1 = a_3 = a_5 = \cdots = 0$ .

## 2.5 L'Hopital's Rule

1. Use L'Hopital's rule to evaluate the following limits.

- |   |   |   |
|---|---|---|
| (a) $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x}$                      | (g) $\lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right)$ | (m) $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}}$                                    |
| (b) $\lim_{x \rightarrow 0} \frac{\sin^2 x}{1 - \cos x}$                  | (h) $\lim_{x \rightarrow 1} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right)$ | (n) $\lim_{x \rightarrow +\infty} \frac{\ln(2x^3 - 5x^2 + 3)}{\ln(4x^2 + x - 7)}$ |
| (c) $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x}$        | (i) $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{\cosh x - 1}$                | (o) $\lim_{x \rightarrow +\infty} x \sin \left( \frac{1}{x} \right)$              |
| (d) $\lim_{x \rightarrow 0} \frac{1 - x \cot x}{x \sin x}$                | (j) $\lim_{x \rightarrow 0} \frac{e - (1+x)^{\frac{1}{x}}}{x}$              | (p) $\lim_{x \rightarrow +\infty} x \left( \frac{\pi}{2} - \tan^{-1} x \right)$   |
| (e) $\lim_{x \rightarrow 0} \frac{\sinh x - \sin x}{x(\cosh x - \cos x)}$ | (k) $\lim_{x \rightarrow 0} \frac{2^x - 1}{x}$                              | (q) $\lim_{x \rightarrow +\infty} x \ln \left( 1 + \frac{3}{x} \right)$           |
| (f) $\lim_{x \rightarrow 0} \frac{\ln \cos 2x}{\ln \cos x}$               | (l) $\lim_{x \rightarrow 0^+} x^{\frac{1}{1+\ln x}}$                        | (r) $\lim_{x \rightarrow +\infty} (e^x + x)^{\frac{1}{x}}$                        |

### 3 Integration

#### 3.1 Fundamental Theorem of Calculus

1. Find  $F'(x)$  for the following functions.

$$(a) F(x) = \int_{\pi}^x \frac{\cos y}{y} dy$$

$$(e) F(x) = \int_{-\sin x}^{\sqrt{\pi}} \cos(y^2) dy$$

$$(b) F(x) = \int_{-\pi}^x e^{\sin 2t} dt$$

$$(f) F(x) = \int_x^{2x} (\ln t)^2 dt$$

$$(c) F(x) = \int_x^1 \sqrt{1+t^2} dt$$

$$(g) F(x) = \int_{x^2}^{x^3} e^{\cos u} du$$

$$(d) F(x) = \int_0^{x^3} e^{u^2} du$$

$$(h) F(x) = \int_{-\sqrt{\ln x}}^{\sqrt{\ln x}} \frac{\sin t}{t} dt$$

#### 3.2 Substitution

Use a suitable substitution to evaluate the following integral.

$$1. \int \frac{dx}{\sqrt{2-5x}}$$

$$11. \int \tan x dx$$

$$2. \int \frac{e^{3x} + 1}{e^x + 1} dx$$

$$12. \int \frac{dx}{1 + e^x}$$

$$3. \int \frac{x}{\sqrt{1-x^2}} dx$$

$$13. \int x(x^2 + 2)^{99} dx$$

$$4. \int x^2 \sqrt[3]{1+x^3} dx$$

$$14. \int \frac{x}{\sqrt{25-x^2}} dx$$

$$5. \int \frac{x dx}{(1+x^2)^2}$$

$$15. \int \frac{x}{\sqrt{3x^2+1}} dx$$

$$6. \int \frac{dx}{\sqrt{x}(1+x)}$$

$$16. \int \frac{x^2}{\sqrt{9-x^3}} dx$$

$$7. \int \frac{1}{x^2} \sin \frac{1}{x} dx$$

$$17. \int x(x+2)^{99} dx$$

$$8. \int x e^{-x^2} dx$$

$$18. \int \frac{x dx}{\sqrt{4x+5}}$$

$$9. \int \frac{e^x dx}{2+e^x}$$

$$19. \int x \sqrt{x-1} dx$$

$$10. \int \frac{dx}{e^x + e^{-x}}$$

$$20. \int (x+2)\sqrt{x-1} dx$$

21.  $\int \frac{x dx}{\sqrt{x+9}}$

22.  $\int_0^1 x^3(1+3x^2)^{\frac{1}{2}} dx$

### 3.3 Integration by Parts

1.  $\int \ln x dx$

13.  $\int x \sin(4x) dx$

2.  $\int x^2 \ln x dx$

14.  $\int x \cos(5-x) dx$

3.  $\int \left(\frac{\ln x}{x}\right)^2 dx$

15.  $\int \cos^{-1}(x) dx$

4.  $\int x e^{-x} dx$

16.  $\int x \cos^{-1}(x) dx$

5.  $\int x^2 e^{-2x} dx$

17.  $\int \tan^{-1}(x) dx$

6.  $\int x \cos x dx$

18.  $\int x \tan^{-1}(x) dx$

7.  $\int x^2 \sin 2x dx$

19.  $\int \ln^2(x) dx$

8.  $\int \sin^{-1} x dx$

20.  $\int x^{99} \ln(x) dx$

9.  $\int x \tan^{-1} x dx$

21.  $\int \frac{\ln(x)}{x^{101}} dx$

10.  $\int \ln(x + \sqrt{1+x^2}) dx$

22.  $\int x \sec^2(x) dx$

11.  $\int x \sin^2 x dx$

23.  $\int e^{2x} \cos(3x) dx$

12.  $\int \sin(\ln x) dx$

### 3.4 Reduction Formula

Prove the following reduction formulas.

1.  $I_n = \int x^n e^{ax} dx; I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}, n \geq 1$

2.  $I_n = \int \sin^n x dx; I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, n \geq 2$

$$3. I_n = \int \cos^n x dx; I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, n \geq 2$$

$$4. I_n = \int \frac{1}{\sin^n x} dx; I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}, n \geq 2$$

$$5. I_n = \int x^n \cos x dx; I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}, n \geq 2$$

$$6. I_n = \int \frac{dx}{(x^2 - a^2)^n}; I_n = -\frac{x}{2a^2(n-1)(x^2 - a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}, n \geq 1$$

$$7. I_n = \int \frac{x^n dx}{\sqrt{x+a}}; I_n = \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2an}{2n+1} I_{n-1}, n \geq 1$$

$$8. I_n = \int (\ln x)^n dx; I_n = x(\ln x)^n - nI_{n-1}, n \geq 1.$$

$$9. I_n = \int_0^1 x^n \sqrt{1-x} dx; I_n = \frac{2n}{2n-3} I_{n-1}, n \geq 2.$$

### 3.5 Trigonometric Integrals

Evaluate

$$1. \int \frac{dx}{1 - \cos x}$$

$$9. \int \tan^5 x dx$$

$$2. \int \sin^5 x \cos x dx$$

$$10. \int \frac{dx}{\sin^4 x \cos^4 x}, dx$$

$$3. \int \sin 3x \sin 5x dx$$

$$11. \int \sin 5x \cos x dx$$

$$4. \int \cos \frac{x}{2} \cos \frac{x}{3} dx$$

$$12. \int \cos x \cos 2x \cos 3x dx$$

$$5. \int \cos^3 x dx$$

$$13. \int \cos^5 x \sin^3 x dx$$

$$6. \int \sin^4 x dx$$

$$14. \int \cos^5 x \sin^4 x dx$$

$$7. \int \frac{dx}{\cos x \sin^2 x}$$

$$15. \int \sin^2 x \cos^4 x dx$$

$$8. \int \frac{\sin x \cos^3 x}{1 + \cos^2 x}, dx$$

### 3.6 Trigonometric Substitution

Evaluate the following integrals by trigonometric substitution.

1.  $\int \frac{x^2}{1+x^2} dx$

2.  $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$

3.  $\int \sqrt{\frac{1+x}{1-x}} dx$

4.  $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

5.  $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

6.  $\int \frac{dx}{\sqrt{4+x^2}}$

7.  $\int x^2 \sqrt{16-x^2} dx$

8.  $\int \frac{dx}{x^2 \sqrt{x^2+4}}$

9.  $\int \frac{dx}{(4x^2+1)^{3/2}}$

10.  $\int \frac{1}{(2x-x^2)^{3/2}}$

### 3.7 Rational Functions

Evaluate the following integrals of rational functions.

1.  $\int \frac{x^2 dx}{1-x^2}$

2.  $\int \frac{x^3}{3+x} dx$

3.  $\int \frac{(1+x)^2}{1+x^2} dx$

4.  $\int \frac{dx}{x^2+2x-3}$

5.  $\int \frac{dx}{(x^2-2)(x^2+3)}$

6.  $\int \frac{x^2+1}{(x+1)^2(x-1)} dx$

7.  $\int \frac{x^2}{(x^2-3x+2)^2} dx$

8.  $\int \frac{x^2+5x+4}{x^4+5x^2+4} dx$

9.  $\int \frac{dx}{(x+1)(x^2+1)}$

10.  $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3} dx$

11.  $\int \frac{4-2x}{(x^2+1)(x-1)^2} dx$

12.  $\int \frac{dx}{x(x^2+1)^2}$

13.  $\int \frac{x^2 dx}{(x-1)(x-2)(x-3)}$

14.  $\int \frac{x dx}{x^2(x^2-2x+2)}$

### 3.8 $t$ -method

Use  $t$ -substitution to evaluate the following integrals.

1.  $\int \frac{dx}{\sin^3 x}$

3.  $\int \frac{dx}{\sin x \cos^4 x}$

2.  $\int \frac{dx}{1+\sin x}$

4.  $\int \frac{dx}{2+\sin x}$

5. 
$$\int \frac{1 - \cos x}{3 + \cos x} dx$$

6. 
$$\int \frac{\cos x + 1}{\sin x + \cos x} dx$$

### 3.9 Piecewise Functions

Find  $\int f(x)dx$  for the following functions  $f(x)$ .

1. 
$$f(x) = \begin{cases} 4x - 1, & x < 1 \\ \frac{3}{\sqrt{x}}, & x \geq 1 \end{cases}$$

2. 
$$f(x) = \begin{cases} e^{2x}, & x < 0 \\ \cos^2 x, & x \geq 0 \end{cases}$$

3. 
$$f(x) = |x - 3|$$

4. 
$$f(x) = \frac{1}{1 + |x|}$$

5. 
$$f(x) = 3|x^2 - 4|$$

6. 
$$f(x) = |\ln x|$$

## 4 Further Problems

1. (a) Evaluate  $\lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x}$ .  
 (b) Using (a), evaluate  $\lim_{x \rightarrow 0^+} (1 - \cos x)^{\frac{1}{\ln x}}$ .
2. Let  $f$  and  $g$  be differentiable functions defined on  $\mathbb{R}$  satisfying the following conditions:
  - $f'(x) = g(x)$  for  $x \in \mathbb{R}$ ;
  - $g'(x) = -f(x)$  for  $x \in \mathbb{R}$ ;
  - $f(0) = 0$  and  $g(0) = 1$ .

By differentiating  $h(x) = [f(x) - \sin x]^2 + [g(x) - \cos x]^2$ , show that  $f(x) = \sin x$  and  $g(x) = \cos x$  for  $x \in \mathbb{R}$ .

3. Let  $a > b > 0$  and define

$$f(x) = \begin{cases} \left(\frac{a^x + b^x}{2}\right)^{1/x} & \text{for } x > 0, \\ \sqrt{ab} & \text{for } x = 0. \end{cases}$$

- (a) (i) Evaluate  $\lim_{x \rightarrow 0^+} f(x)$ .  
 Hence show that  $f$  is continuous at  $x = 0$ .  
 (ii) Show that  $\lim_{x \rightarrow \infty} f(x) = a$ .
- (b) Let  $h(t) = (1+t)\ln(1+t) + (1-t)\ln(1-t)$  for  $0 \leq t < 1$  and  $g(x) = \ln f(x)$  for  $x \geq 0$ .  
 (i) Show that  $h(t) > h(0)$  for  $0 < t < 1$ .  
 (ii) For  $x > 0$ , let  $t = \frac{a^x - b^x}{a^x + b^x}$ . Show that  $0 < t < 1$  and

$$h(t) = 2 \left[ \frac{a^x \ln a^x + b^x \ln b^x}{a^x + b^x} + \ln \left( \frac{2}{a^x + b^x} \right) \right].$$

- (iii) Show that for  $x > 0$ ,

$$x^2 g'(x) = \frac{a^x \ln a^x + b^x \ln b^x}{a^x + b^x} + \ln \left( \frac{2}{a^x + b^x} \right).$$

Hence deduce that  $f(x)$  is strictly increasing on  $[0, \infty)$ .

4. Let  $f$  be real-valued function on  $[0, 1]$  and differentiable in  $(0, 1)$ . Suppose  $f$  satisfies
  - $f(0) = 0$ ;
  - $f(1) = 1/2$ ;
  - $0 < f'(t) < 1$  for  $t \in (0, 1)$ .

Define  $F(x) = 2 \int_0^x f(t) dt - [f(x)]^2$  for  $x \in [0, 1]$ .

(a) Show that  $F'(x) > 0$  for  $x \in (0, 1)$ .

(b) Show that  $\int_0^1 f(t) dt > \frac{1}{8}$ .

5. Let  $u : \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function satisfying the following conditions:

- $u''(x) = -u(x)$  for all  $x \in \mathbb{R}$ ;
- $u(0) = 0$ ;
- $u'(0) = 1$ .

Define  $v(x) = u(x) - \sin x$  for all  $x \in \mathbb{R}$ .

By differentiating  $w(x) = [v(x)]^2 + [v'(x)]^2$ , prove that  $u(x) = \sin x$  for all  $x \in \mathbb{R}$ .

6. (a) Let  $f$  be real-valued function defined on an open interval  $I$  and  $f''(x) \geq 0$  for  $x \in I$ .

(i) Let  $a, b, c \in I$  with  $a < c < b$ . Using Mean Value Theorem, show that

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(b) - f(c)}{b - c}.$$

Hence show that

$$f(c) \leq \frac{b - c}{b - a} f(a) + \frac{c - a}{b - a} f(b).$$

(ii) Let  $a, b \in I$  with  $a < b$  and  $\lambda \in (0, 1)$ , show that

$$a < \lambda a + (1 - \lambda)b < b.$$

Hence show that  $f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)$ .

(b) Let  $0 < a < b$ . Using (a)(ii), show that

(i) if  $p > 1$  and  $0 < \lambda < 1$ , then

$$[\lambda a + (1 - \lambda)b]^p \leq \lambda a^p + (1 - \lambda)b^p;$$

(ii) if  $0 < \lambda < 1$ , then  $\lambda a + (1 - \lambda)b \geq a^\lambda b^{1-\lambda}$ .

7. For any real number  $x$ , let  $[x]$  denote the greatest integer not greater than  $x$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} \frac{1}{2} & \text{when } x \text{ is an integer,} \\ x - [x] - \frac{1}{2} & \text{when } x \text{ is not an integer.} \end{cases}$$

(a) (i) Prove that  $f$  is a periodic function with period 1.

(ii) Sketch the graph of  $f(x)$ , where  $-2 \leq x \leq 3$ .

- (iii) Write down all the real number(s)  $x$  at which  $f$  is discontinuous.
- (b) Define  $F(x) = \int_0^x f(t) dt$  for all real numbers  $x$ .
- (i) If  $0 \leq x \leq 1$ , prove that  $F(x) = \frac{x^2 - x}{2}$ .
- (ii) Is  $F$  a periodic function? Explain your answer.
- (iii) Evaluate  $\int_0^\pi F(x) dx$ .
8. (a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function. Assume that  $a$  and  $b$  are two distinct real numbers.

- (i) Find a constant  $k$  (independent of  $x$ ) such that the function

$$h(x) = f(x) - f(b) - f'(x)(x - b) - k(x - b)^2$$

satisfies  $h(a) = 0$ . Also find  $h(b)$ .

- (ii) Let  $I$  be the open interval with end points  $a$  and  $b$ . Using Mean Value Theorem and (a)(i), prove that there exists a real number  $c \in I$  such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2.$$

- (b) Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function. Assume that there exists a real number  $\beta \in (0, 1)$  such that  $g(x) \leq g(\beta) = 1$  for all  $x \in (0, 1)$ .

- (i) Using (a)(ii), prove that there exists a real number  $\gamma \in (0, 1)$  such that

$$g(1) = 1 + \frac{g''(\gamma)}{2}(1 - \beta)^2.$$

- (ii) If  $g''(x) \geq -2$  for all  $x \in (0, 1)$ , prove that  $g(0) + g(1) \geq 1$ .

9. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying the following conditions:

- $f(x + y) = e^x f(y) + e^y f(x)$  for all  $x, y \in \mathbb{R}$ ;
- $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 2005$ .

- (a) Find  $f(0)$ .

- (b) Find  $\lim_{h \rightarrow 0} f(h)$ . Hence prove that  $f$  is a continuous function.

- (c) (i) Prove that  $f$  is differentiable everywhere and that

$$f'(x) = 2005e^x + f(x)$$

for all  $x \in \mathbb{R}$ .

- (ii) Let  $n$  be a positive integer. Using (c)(i), find  $f^{(n)}(0)$ .

- (d) By considering the derivative of the function  $\frac{f(x)}{e^x}$ , find  $f(x)$ .

10. Let  $I_m = \int_0^{\pi/2} \cos^m t dt$  where  $m = 0, 1, 2, \dots$

- (a) (i) Evaluate  $I_0$  and  $I_1$ .  
(ii) Show that  $I_m = \frac{m-1}{m} I_{m-2}$  for  $m \geq 2$ .  
Hence, evaluate  $I_{2n}$  and  $I_{2n+1}$  for  $n \geq 1$ .
- (b) Show that  $I_{2n-1} \geq I_{2n} \geq I_{2n+1}$  for  $n \geq 1$ .
- (c) Let  $A_n = \frac{1}{2n+1} \left[ \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right]^2$  where  $n = 0, 1, 2, \dots$
- (i) Using (a) and (b), show that  $\frac{2n+1}{2n} A_n \geq \frac{\pi}{2} \geq A_n$ .  
(ii) Show that  $\{A_n\}$  is monotonic increasing.  
(iii) Evaluate  $\lim_{n \rightarrow \infty} \frac{1}{2n+1} \left[ \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right]^2$ .

11. (a) Let  $f$  be a non-negative continuous function on  $[a, b]$ . Define

$$F(x) = \int_a^x f(t) dt$$

for  $x \in [a, b]$ .

Show that  $F$  is an increasing function on  $[a, b]$ .

Hence deduce that if  $\int_a^b f(t) dt = 0$ , then  $f(x) = 0$  for all  $x \in [a, b]$ .

- (b) Let  $g$  be a continuous function on  $[a, b]$ . If  $\int_a^b g(x)u(x) dx = 0$  for any continuous function  $u$  on  $[a, b]$ , show that  $g(x) = 0$  for all  $x \in [a, b]$ .
- (c) Let  $h$  be a continuous function on  $[a, b]$ . Define

$$A = \frac{1}{b-a} \int_a^b h(t) dt.$$

- (i) If  $v(x) = h(x) - A$  for all  $x \in [a, b]$ , show that  $\int_a^b v(x) dx = 0$ .
- (ii) If  $\int_a^b h(x)w(x) dx = 0$  for any continuous function  $w$  on  $[a, b]$  satisfying  $\int_a^b w(x) dx = 0$ , show that  $h(x) = A$  for all  $x \in [a, b]$ .

12. Let  $n$  be a positive integer.

- (a) Show that  $\frac{1}{1-t^2} = (1+t+t^2+\cdots+t^{2n-2}) + \frac{t^{2n}}{1-t^2}$  for  $t^2 \neq 1$ .
- (b) For  $-1 < x < 1$ , show that
- (i)  $\int_0^x \frac{t}{1-t^2} dt = \ln \frac{1}{\sqrt{1-x^2}}$ .
- (ii)  $\int_0^x \frac{t^{2n+1}}{1-t^2} dt = \ln \frac{1}{\sqrt{1-x^2}} - \left( \frac{x^2}{2} + \frac{x^4}{4} + \cdots + \frac{x^{2n}}{2n} \right)$ .

(c) Show that  $0 \leq \ln 3 - \sum_{k=1}^n \frac{1}{2k} \left(\frac{8}{9}\right)^k \leq \frac{9}{2n+2} \left(\frac{8}{9}\right)^{n+1}$ .

Hence evaluate  $\sum_{k=1}^{\infty} \frac{1}{2k}$ .

13. (a) Let  $f : (-1, 1) \rightarrow \mathbb{R}$  be a function with derivatives of any order. For each  $m = 1, 2, 3, \dots$  and  $x \in (-1, 1)$ , define

$$I_m = \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f^{(m)}(t) dt.$$

(i) Prove that  $I_{m+1} = I_m - \frac{f^{(m)}(0)}{m!} x^m$ .

- (ii) Using (a)(i), prove that

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} x^k + I_m,$$

where  $f^{(0)} = f$ .

- (b) Define  $g(x) = \frac{1}{\sqrt{1-x^2}}$  for all  $x \in (-1, 1)$ . Let  $n$  be a positive integer.

- (i) Prove that

$$(1-x^2)g'(x) - xg(x) = 0.$$

Hence deduce that

$$(1-x^2)g^{(n+1)}(x) - (2n+1)xg^{(n)}(x) - n^2g^{(n-1)}(x) = 0,$$

where  $g^{(0)}(x) = g(x)$ .

- (ii) Prove that  $g^{(2n-1)}(0) = 0$  and  $g^{(2n)}(0) = \left(\frac{(2n)!}{(2^n)(n!)}\right)^2$ .

- (iii) Using (a), prove that

$$g(x) = \sum_{k=0}^{n-1} \frac{C_k^{2k}}{2^{2k}} x^{2k} + \frac{1}{(2n-1)!} \int_0^x (x-t)^{2n-1} g^{(2n)}(t) dt.$$

## 5 Answers

### 1. Preliminaries

#### Section 1.1: Functions

1.  $x < -4$  or  $x > 2$
2. Not a graph of a function, because there are values of  $x$  that correspond to more than one value of  $y$ .
3.  $x = 0$  or  $x = 1$
4. (a) False  
(b) False  
(c) False  
(d) False  
(e) False  
(f) False  
(g) False

#### Section 1.2: Limits and Derivatives

1. (a)  $\frac{1}{24}$                       (c)  $\frac{1}{4}$                       (e)  $\frac{1}{8}$   
(b) 4                              (d)  $\frac{1}{e^2}$
2. (a)  $3x^2 - 4$                       (c)  $-\frac{3}{x^3}$   
(b)  $\frac{1}{\sqrt{x}}$                               (d)  $-\frac{1}{2x^{\frac{3}{2}}}$
3. (a)  $f'(x) = \frac{\pi x^{\pi-1}}{x^\pi + e^\pi}$   
(b)  $g'(x) = -\frac{\pi x \ln \pi + 4x^3}{2(\pi x + x^4)^{3/2}}$   
(c)  $h'(x) = e^{1/x^2} - \frac{2e^{1/x^2}}{x^2} + \frac{1 - \ln(2x)}{2x^{3/2}}$   
(d)  $u'(x) = \frac{4e^{-x} - 2xe^{-x} + 5\sqrt{x}}{4(e^{-x} + \sqrt{x})^{3/2}}$   
(e)  $v'(x) = \frac{174x}{1+x^2} - \frac{2}{x}$
4.  $\frac{dy}{dx} = \frac{1}{\ln[(x+1)(x+2)^2(x+3)^3]} \left( \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} \right)$
5. 689
6. (a) False

- (b) False
- (c) False
- (d) False
- (e) False

### Section 1.3: Applications of Derivatives

1.  $(-1, 4)$  or  $(3, 12)$
2.  $a = -3, b = 3, c = 0$
3.  $\frac{1}{9\pi}$  meters per hour
4. (a) The equation of the tangent  $T_P$  to  $C$  at  $P$  is  $y = 2$ .  
The equation of the normal  $N_P$  to  $C$  at  $P$  is  $x = 1$ .  
(b) The equation of the tangent  $T_Q$  to  $C$  at  $Q$  is  $y = -3x + 4$ .  
 $(x, y) = (0, 4)$  satisfies the equation of  $T_Q$ .
5. (a)  $V = \frac{\pi h^3}{9}$   
(b) The water level is falling at a rate of  $\frac{3}{16}$  cm per second when the depth of water is 4 cm.
6. (a) The  $y$ -intercept of  $C$  is  $-\frac{4}{3}$ .  
The  $x$ -intercepts of  $C$  are  $-\frac{2}{3}, 2$ .  
(b)  $\frac{dy}{dx} = 1 - \frac{1}{(1+x)^2}$ .  
 $\frac{d^2y}{dx^2} = \frac{2}{(1+x)^3}$ .  
(c) The turning points of  $C$  are  $\left(-2, -\frac{16}{3}\right)$  and  $\left(0, -\frac{4}{3}\right)$ .  
 $\left(-2, -\frac{16}{3}\right)$  is a relative maximum.  
 $\left(0, -\frac{4}{3}\right)$  is a relative minimum.  
(d) The asymptotes of the curve  $C$  are the lines  $x = -1, y = x - \frac{7}{3}$   
(e)
7. The function  $f(x) = \frac{x^2 + kx + 9}{x^2 + 1}$ , where  $k$  is a constant, attains a stationary value at  $x = 3$ .

- (a)  $k = -6$ .
- (b) i. The  $x$ -intercept is 3. The  $y$ -intercept is 9.  
 ii. The turning points of the curve  $C$  are  $\left(-\frac{1}{3}, 10\right)$ ,  $(3, 0)$ .  
 $\left(-\frac{1}{3}, 10\right)$  is a relative maximum.  $(3, 0)$  is a relative minimum.  
 iii. The only asymptote of  $C$  is  $y = 1$ .
- (c)
8. Let  $\Sigma$  be a sphere with (fixed) radius  $R$ .
- (a) Suppose  $\Gamma$  be a right circular cylinder inscribed in the sphere  $\Sigma$ , with height  $h$ , radius  $r$ , volume  $V$  and surface area  $A$ .  
 i.  $V = 2\pi r^2\sqrt{R^2 - r^2}$ .  
 ii.  $A = 2\pi r^2 + 4\pi r\sqrt{R^2 - r^2}$ .
- (b) The largest possible value of  $V$  is  $\frac{4\pi R^3}{3\sqrt{3}}$ .
- (c) The largest possible value of  $A$  is  $3\pi R^2$ .
9. Let  $\beta \in (1, +\infty)$ . Let  $f : (0, +\infty) \rightarrow \mathbb{R}$  be the function defined by  $f(x) = x^\beta + \beta - 1 - \beta x$  for any  $x \in (0, +\infty)$ .
- (a) i.  $f'(x) = \beta x^{\beta-1} - \beta$  for any  $x \in (0, +\infty)$ .  
 ii. Suppose  $0 < x < 1$ . Then  $0 < x^{\beta-1} < 1$ . Therefore  $f'(x) = \beta(x^{\beta-1} - 1) < 0$ .  
 Hence  $f$  is strictly decreasing on  $(0, 1]$ .  
 iii. Suppose  $x > 1$ . Then  $x^{\beta-1} > 1$ . Therefore  $f'(x) = \beta(x^{\beta-1} - 1) > 0$ .  
 Hence  $f$  is strictly increasing on  $[1, +\infty)$ .  
 iv.  $f$  attains the the minimum at 1 on  $(0, +\infty)$ , with  $f(1) = 0$ .
- (b) Let  $r \in (-1, +\infty)$ .  $1+r \in (0, +\infty)$ . Then  $f(1+r) = (1+r)^\beta + \beta - 1 - \beta(1+r) \geq 0$ .  
 Therefore  $(1+r)^\beta \geq 1 + \beta r$ .

### Section 1.4: Integration

1. (a)  $27x - 9x^3 + \frac{9}{5}x^5 - \frac{1}{7}x^7 + C$  (c)  $\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + C$   
 (b)  $\frac{625}{3}x^3 - 125x^4 + 30x^5 - \frac{10}{3}x^6 + \frac{1}{7}x^7 + C$  (d)  $4t^2 - \frac{8}{3}t^{\frac{3}{4}} + C$
2. (a)  $-\frac{2}{5}\sqrt{2-5x} + C$  (e)  $-\frac{1}{2(1+x^2)} + C$   
 (b)  $\frac{1}{2}e^{2x} - e^x + x + C$  (f)  $-\frac{1}{2}e^{-x^2} + C$   
 (c)  $-\sqrt{1-x^2} + C$  (g)  $\ln(2+e^x) + C$   
 (d)  $\frac{1}{4}(1+x^3)^{\frac{4}{3}} + C$  (h)  $x - \ln(1+e^x) + C$
- 3.



## 2. Differentiation

### Section 2.2: Limits and Continuity

- |                   |                   |                          |
|-------------------|-------------------|--------------------------|
| 1. (a) 1          | (e) 3             | (i) $\frac{1}{2}$        |
| (b) $\frac{3}{4}$ | (f) 4             | (j) $\frac{1}{\sqrt{3}}$ |
| (c) 2             | (g) $\frac{1}{2}$ | (k) 1                    |
| (d) 2             | (h) $\frac{1}{4}$ | (l) 1                    |
| 2. (a) 0          | (g) $-2$          | (m) $\frac{1}{3}$        |
| (b) $\frac{3}{5}$ | (h) $-1$          | (n) 3                    |
| (c) 3             | (i) 3             | (o) 3                    |
| (d) $\frac{2}{3}$ | (j) $-2$          | (p) 0                    |
| (e) 3             | (k) 0             | (q) 0                    |
| (f) $\frac{5}{4}$ | (l) 1             | (r) 0                    |

### Section 2.3: Derivatives

- |                        |          |                        |
|------------------------|----------|------------------------|
| 1. (a) 0               | (c) 0    | (e) not differentiable |
| (b) not differentiable | (d) $-2$ | (f) 0                  |

2.  $a = -1, b = \pi$

3.  $a = 0, b = \frac{1}{2}$

- |  |  |
|--|--|
| 4. (a) $3x^2 - 4$                          | (n) $-2x \sin(x^2)$                                  |
| (b) $(x - 1)/2x^{3/2}$                     | (o) $(3x^3 + x^2 + 1)e^{x^3+x}$                      |
| (c) $(5x^2 + 2x)e^{5x}$                    | (p) $1/(x \ln x)$                                    |
| (d) $-\sin x \ln x + \cos x/x$             | (q) $\cos(x)e^{\sin x}$                              |
| (e) $\cos 2x$                              | (r) $1/(x^2 + 1)^{3/2}$                              |
| (f) $(3 \sin x - 1)/\cos^2 x$              | (s) $1/\sqrt{x^2 + 1}$                               |
| (g) $\cot x - x \csc^2 x$                  | (t) $(1 + 2\sqrt{x})/(4\sqrt{x}\sqrt{x + \sqrt{x}})$ |
| (h) $10/(x + 2)^2$                         | (u) $\sinh 2x$                                       |
| (i) $1 - 2/(x + 1)^2$                      | (v) $\frac{\sinh x}{\cosh^2 x} + \sinh x$            |
| (j) $(x \cos(x) - \sin x)/x^2$             | (w) $\coth x$  |
| (k) $(2x \tan^2 x - \tan x + 2x)/2x^{3/2}$ | (x) $1/\left(2\sqrt{x(1-x)}\right)$                  |
| (l) $14x(x^2 + 1)^6$                       | (y) $-x/(x^2 + 1)^{3/2}$                             |
| (m) $2x^3/\sqrt{x^4 + 1}$                  |  |

5.

- (a)  $3^x \ln 3$  (d)  $x^{\sqrt{x}-1/2}(\ln(x)/2 + 1)$   
 (b)  $-\ln 2 \sin(x)2^{\cos(x)}$  (e)  $(\cos x)^{\sin x-1}(\cos^2 x + \ln(\cos x) \cos^2 x - 1)$   
 (c)  $(\ln(x) + 1)x^x$  (f)  $x^x x^{x^x} (x(\ln x)^2 + x \ln x + 1)/x$
6. (a)  $-\frac{x}{y}$  (d)  $-\frac{1}{x^2}$   
 (b)  $-\frac{y^2 + 3x^2y}{x^3 + 2xy}$  (e)  $\frac{2x + 2y}{x \cos(xy) - 2y - 2x}$   
 (c)  $\frac{3x^2 - 2y}{2x - 3y^2}$  (f)  $\frac{\frac{y \sin(y/x)}{x^2} - \frac{1}{x+y}}{\frac{1}{x+y} + \frac{\sin(y/x)}{x}}$
7. (a)  $x^{-\frac{3}{2}}e^{x^2} + x^{\frac{1}{2}}e^{x^2} + 4x^{\frac{5}{2}}e^{x^2}$  (f)  $-\frac{8x^2 + 6y^3}{9y^5}$   
 (b)  $-3x(1 + x^2)^{-\frac{5}{2}}$  (g)  $\frac{10(2x^6y + 3x^4y^3 - 81y^7)}{x^2(x^2 - 9y^2)^3}$   
 (c)  $-\frac{2}{x^2} \ln x + \frac{2}{x^2}$  (h)  $2 \cdot 3^{x^2} \ln 3 + 4 \cdot x^2 3^{x^2} (\ln 3)^2$   
 (d)  $\frac{\cos^3 x + 2 \sin^2 x \cos x}{\cos^4 x}$  (i)  $\frac{4y}{x^2}(\ln x)^2 + \frac{2y}{x^2}(1 - \ln x)$   
 (e)  $\frac{-2x}{(1 + x^2)^2}$

8. Prove that the Chebyshev polynomials

$$T_m(x) = \frac{1}{2^{m-1}} \cos(m \cos^{-1} x), \quad m = 0, 1, 2, \dots$$

satisfy

$$(1 - x^2)T_m''(x) - xT_m'(x) + m^2T_m(x) = 0$$

*Proof.* By direct computations,

$$T_m'(x) = \frac{m \sin(m \cos^{-1} x)}{2^{m-1} \sqrt{1 - x^2}}$$

and

$$T_m''(x) = \frac{m}{2^{m-1}} \left( \frac{x \sin(m \cos^{-1} x)}{(1 - x^2)^{\frac{3}{2}}} - \frac{m}{1 - x^2} \cos(m \cos^{-1} x) \right).$$

Hence,

$$\begin{aligned} & (1 - x^2)T_m''(x) - xT_m'(x) + m^2T_m(x) \\ &= (1 - x^2) \frac{m}{2^{m-1}} \left( \frac{x \sin(m \cos^{-1} x)}{(1 - x^2)^{\frac{3}{2}}} - \frac{m}{1 - x^2} \cos(m \cos^{-1} x) \right) \\ & \quad - x \frac{m \sin(m \cos^{-1} x)}{2^{m-1} \sqrt{1 - x^2}} + m^2 \frac{1}{2^{m-1}} \cos(m \cos^{-1} x) \\ &= 0. \end{aligned}$$

□

9. Prove that the Legendre polynomials

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m \quad m = 0, 1, 2, \dots$$

satisfy

$$(1 - x^2)P_m''(x) - 2xP_m'(x) + m(m + 1)P_m(x) = 0$$

*Proof.* Let  $g(x) = (x^2 - 1)^m$ , then  $P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} g(x)$ .

Because

$$-\frac{d}{dx}(x^2 - 1)^m = 2mx(x^2 - 1)^{m-1},$$

therefore,

$$-\frac{d(x^2 - 1)^m}{dx}(x^2 - 1) + 2mx(x^2 - 1)^m = 0.$$

We get

$$\begin{aligned} & -g'(x)(x^2 - 1) + 2mxg(x) = 0 \\ \Rightarrow & \frac{d^{m+1}}{dx^{m+1}} (-g'(x)(x^2 - 1) + 2mxg(x)) = 0. \end{aligned}$$

Apply Leibniz's rule,

$$\begin{aligned} \Rightarrow & -\left(g^{(m+2)}(x)(x^2 - 1) + C_1^{m+1}g^{(m+1)}(x) \cdot 2x + C_2^{m+1}g^{(m)}(x) \cdot 2\right) \\ & \quad + 2m\left(g^{(m+1)}(x) \cdot x + C_1^{m+1}g^{(m)}(x)\right) = 0 \\ \Rightarrow & (1 - x^2)g^{(m+2)}(x) - 2xg^{(m+1)}(x) + m(m + 1)g^{(m)}(x) = 0 \\ \Rightarrow & \frac{1}{2^m m!} \left((1 - x^2)g^{(m+2)}(x) - 2xg^{(m+1)}(x) + m(m + 1)g^{(m)}(x)\right) = 0 \\ \Rightarrow & (1 - x^2)P_m''(x) - 2xP_m'(x) + m(m + 1)P_m(x) = 0. \end{aligned}$$

□

10.

11.

## Section 2.4: Mean Value Theorem and Taylor's Theorem

1. Let  $f(x) = x^p$  and apply the mean value theorem on  $[x, y]$ , there exists  $c \in (x, y)$  such that

$$\frac{x^p - y^p}{x - y} = pc^{p-1}.$$

Note that  $y < c < x$  and  $p > 1$ , so  $y^{p-1} < c^{p-1} < x^{p-1}$ . Therefore,

$$py^{p-1} < \frac{x^p - y^p}{x - y} < px^{p-1}.$$

The result follows by multiplying  $(x - y)$ .

2. Applying mean value theorem, there exist  $a$  and  $b$  such that  $x_1 < a < x_2 < b < x_3$  and

$$\frac{\sin x_2 - \sin x_1}{x_2 - x_1} = \cos a \quad \text{and} \quad \frac{\sin x_3 - \sin x_2}{x_3 - x_2} = \cos b.$$

Since  $\cos x$  is strictly decreasing on  $[0, \pi]$ ,  $\cos a > \cos b$  and the result follows.

3. Let  $f(x) = \ln(1+x)$ , for  $x > 0$ , then  $f'(x) = \frac{1}{1+x}$ . By mean value theorem, there exists  $c \in (0, x)$  such that

$$\begin{aligned} \frac{\ln(1+x) - \ln 1}{(1+x) - 1} &= \frac{1}{1+c} \\ \ln(1+x) &= \frac{x}{1+c} \end{aligned}$$

Since  $0 < c < x$ , we have  $\frac{x}{1+x} < \frac{x}{1+c} < x$ . Hence

$$\frac{x}{1+x} < \ln(1+x) < x.$$

Since the above inequality holds for all  $x > 0$ , we can replace  $x$  by  $\frac{1}{x}$  to obtain

$$\frac{1/x}{1+(1/x)} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}. \text{ Therefore,}$$

$$\frac{1}{1+x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}.$$

4. Let  $f(x) = \frac{a_2}{2}x^2 + \frac{a_3}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1}$ . Applying mean value theorem, there exists  $c \in (0, 1)$  such that

$$\begin{aligned} \frac{f(1) - f(0)}{1 - 0} &= f'(c) \\ \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} &= a_1c + a_2c^2 + \cdots + a_nc^n. \end{aligned}$$

Therefore, the equation has a root  $c$  in  $(0, 1)$ .

5. Applying the mean value theorem on  $[0, a]$  and  $[b, a+b]$ , there exist  $p \in (0, a)$  and  $q \in (b, a+b)$  such that

$$\frac{f(a) - f(0)}{a - 0} = f'(p) \quad \text{and} \quad \frac{f(a+b) - f(b)}{(a+b) - b} = f'(q).$$

Since  $f'(x)$  is monotonic increasing on  $(0, \infty)$ ,  $f'(p) \leq f'(q)$ . Therefore,

$$\begin{aligned} \frac{f(a)}{a} &\leq \frac{f(a+b) - f(b)}{a} \\ f(a) &\leq f(a+b) - f(b) \\ f(a) + f(b) &\leq f(a+b) \end{aligned}$$

6. Let  $x, y \in \mathbb{R}$ . The inequality is automatically satisfied if  $x = y$ , so we only need to consider  $x \neq y$ . Now, suppose  $x > y$ . Applying the mean value theorem on  $[y, x]$ , there exists  $c \in (y, x)$  such that

$$\frac{\sin x - \sin y}{x - y} = \cos c \leq 1.$$

Therefore,  $|\sin x - \sin y| \leq |x - y|$ . Similarly, it holds for the case  $y < x$ . Thus,  $\sin x$  satisfies the Lipschitz condition on  $\mathbb{R}$  by taking  $L = 1$ .

7. (a) Applying the mean value theorem on  $[k, k + 1]$ , there exists  $c \in (k, k + 1)$  such that

$$\frac{f(k + 1) - f(k)}{(k + 1) - k} = f'(c).$$

Note that,  $k < c < k + 1$  and  $f'(x)$  is strictly decreasing for  $x > 0$ , so  $f'(k + 1) < f'(c) < f'(k)$ . Therefore,

$$f'(k + 1) < f(k + 1) - f(k) < f'(k).$$

- (b) By the result in (a), for  $k = 1, 2, \dots, n$ ,

$$f'(k + 1) < f(k + 1) - f(k) < f'(k).$$

Therefore,  $\sum_{k=1}^n f'(k + 1) < \sum_{k=1}^n (f(k + 1) - f(k)) < \sum_{k=1}^n f'(k)$ . Simplify this inequality, we have the result

$$f'(2) + f'(3) + \dots + f'(n) < f(n) - f(1) < f'(1) + f'(2) + \dots + f'(n - 1).$$

8. (a)  $x - \frac{x^2}{2} + \frac{x^3}{3}$

(b)  $1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}$

(c)  $x + \frac{x^3}{6} + \frac{x^5}{120}$

9. (a)  $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots$

(b)  $\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n!} (x - \frac{\pi}{2})^{2n} = 1 - \frac{1}{2!} (x - \frac{\pi}{2})^2 + \frac{1}{4!} (x - \frac{\pi}{2})^4 - \dots$

(c)  $\sum_{n=0}^{\infty} \frac{e}{n!} (x - 1)^n = e + e(x - 1) + \frac{e}{2!} (x - 1)^2 + \dots$

10. (a)  $P_3(x) = x + \frac{x^2}{2} + \frac{x^3}{3}$

(b)  $\ln 0.99 = f(0.01) \approx P_3(0.01) = 0.01005033$

11. (a) Firstly, since  $f$  is odd,  $-f(0) = f(-0)$ . Therefore,  $f(0) = 0$ . Then we have,

$$\begin{aligned} -f(x) &= f(-x) \\ -f'(x) &= -f'(-x) \\ f'(x) &= f'(-x) \\ f''(x) &= -f''(-x) \\ -f''(x) &= f''(-x) \end{aligned}$$

As we can see,  $f'$  is an even function and  $f''$  is an odd function again. By repeating the above, it can be shown that  $f^{(2n)}$  is an odd function and so  $f^{(2n)}(0) = 0$  for all positive integers  $n$ . The result follows as  $a_{2n} = \frac{f^{(2n)}(0)}{(2n)!} = 0$ .

- (b) Similar to (a), what we have to show is  $f^{(2n-1)}(0) = 0$  for all positive integers  $n$ .

### Section 2.5: L'Hopital's Rule

- |                      |                   |                   |
|----------------------|-------------------|-------------------|
| 1. (a) $\frac{3}{5}$ | (g) $\frac{1}{2}$ | (m) $\frac{1}{e}$ |
| (b) 2                | (h) $\frac{1}{2}$ | (n) $\frac{3}{2}$ |
| (c) 6                | (i) 1             | (o) 1             |
| (d) $\frac{1}{3}$    | (j) $\frac{e}{2}$ | (p) 1             |
| (e) $\frac{1}{3}$    | (k) $\ln 2$       | (q) 3             |
| (f) 4                | (l) $e$           | (r) $e$           |

### 3. Integration

#### Section 3.1: Fundamental Theorem of Calculus

- |                           |                               |   |
|---------------------------|-------------------------------|---|
| 1. (a) $\frac{\cos x}{x}$ | (d) $3x^2e^{x^6}$             | (g) $3x^2e^{\cos(x^3)} - 2xe^{\cos(x^2)}$ |
| (b) $e^{\sin 2x}$         | (e) $\cos x \cos(\sin^2 x)$   | (h) $\frac{\sin \sqrt{x}}{x\sqrt{\ln x}}$ |
| (c) $-\sqrt{1+x^2}$       | (f) $2(\ln 2x)^2 - (\ln x)^2$ |   |

#### Section 3.2: Substitution

- |   |  |
|---|--|
| 1. $-\frac{2}{5}\sqrt{2-5x} + C$          | 12. $x - \ln(1 + e^x) + C$                                 |
| 2. $\frac{1}{2}e^{2x} - e^x + x + C$      | 13. $\frac{1}{200}(x^2 + 2)^{100} + C$                     |
| 3. $-\sqrt{1-x^2} + C$                    | 14. $-\sqrt{25-x^2} + C$                                   |
| 4. $\frac{1}{4}(1+x^3)^{\frac{4}{3}} + C$ | 15. $\frac{1}{3}\sqrt{3x^2+1} + C$                         |
| 5. $-\frac{1}{2(1+x^2)} + C$              | 16. $-\frac{2}{3}\sqrt{9-x^3} + C$                         |
| 6. $2 \tan^{-1} \sqrt{x} + C$             | 17. $\frac{(x+2)^{101}}{101} - \frac{(x+2)^{100}}{50} + C$ |
| 7. $\cos \frac{1}{x} + C$                 | 18. $\frac{1}{12}(2x-5)\sqrt{4x+5}$                        |
| 8. $-\frac{1}{2}e^{-x^2} + C$             | 19. $\frac{2}{15}(x-1)^{3/2}(3x+2)$                        |
| 9. $\ln(2+e^x) + C$                       | 20. $\frac{2}{5}(x-1)^{3/2}(x+4)$                          |
| 10. $\tan^{-1} e^x + C$                   | 21. $\frac{2}{3}(x-18)\sqrt{x+9}$                          |
| 11. $-\ln  \cos x  + C$                   | 22. $\frac{1}{135}(3x^2+1)^{3/2}(9x^2-2)$                  |

#### Section 3.3: Integration by Parts

- |  |  |
|--|--|
| 1. $x \ln x - x + C$                                     | 10. $x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C$                               |
| 2. $\frac{x^3}{3}(\ln x - \frac{1}{3}) + C$              | 11. $\frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + C$            |
| 3. $-\frac{1}{x}((\ln x)^2 + 2 \ln x + 2) + C$           | 12. $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$                               |
| 4. $-(x+1)e^{-x} + C$                                    | 13. $\frac{1}{16} \sin(4x) - \frac{1}{4}x \cos(4x) + C$                        |
| 5. $-\frac{e^{-2x}}{4}(2x^2 + 2x + 1) + C$               | 14. $\cos(5-x) - x \sin(5-x) + C$  |
| 6. $x \sin x + \cos x + C$                               | 15. $x \cos^{-1}(x) - \sqrt{1-x^2} + C$  |
| 7. $-\frac{2x^2-1}{4} \cos 2x + \frac{x}{2} \sin 2x + C$ | 16. $\frac{1}{4}(-x\sqrt{1-x^2} + 2x^2 \cos^{-1}(x) + \sin^{-1}(x)) + C$       |
| 8. $x \sin^{-1} x + \sqrt{1-x^2} + C$                    | 17. $x \tan^{-1}(x) - \frac{1}{2} \log(x^2 + 1) + C$                           |
| 9. $-\frac{x}{2} + \frac{1+x^2}{2} \tan^{-1} x + C$      | 18. $\frac{1}{2}x^2 \tan^{-1}(x) - \frac{x}{2} + \frac{1}{2} \tan^{-1}(x) + C$ |

19.  $2x + x \ln^2(x) - 2x \ln(x) + C$
20.  $\frac{1}{100}x^{100} \log(x) - \frac{x^{100}}{10000} + C$
21.  $-\frac{1}{10000x^{100}} - \frac{\log(x)}{100x^{100}} + C$
22.  $x \tan(x) + \ln(\cos(x)) + C$
23.  $\frac{1}{13}e^{2x}(3 \sin(3x) + 2 \cos(3x)) + C$

### Section 3.5: Trigonometric Integrals

1.  $-\cot \frac{x}{2} + C$
2.  $\frac{1}{6} \sin^6 x + C$
3.  $\frac{1}{4} \sin 2x - \frac{1}{16} \sin 8x + C$
4.  $3 \sin \frac{x}{6} + \frac{3}{5} \sin \frac{5x}{6} + C$
5.  $\sin x - \frac{1}{3} \sin^3 x + C$
6.  $\frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$
7.  $-\frac{1}{\sin x} + \frac{1}{2} \ln \frac{1+\sin x}{1-\sin x} + C$
8.  $-\frac{1}{2} \cos^2 x + \frac{1}{2} \ln(1 + \cos^2 x) + C$
9.  $\frac{\tan^4}{4} - \frac{\tan^2 x}{2} - \ln |\cos x| + C$
10.  $-8 \cot 2x - \frac{8}{3} \cot^3 2x + C$
11.  $-\frac{1}{8} \cos 4x - \frac{1}{12} \cos 6x + C$
12.  $\frac{x}{4} + \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24} + C$
13.  $\frac{\cos^8(x)}{8} - \frac{\cos^6(x)}{6} + C$
14.  $\frac{\sin^9(x)}{9} - \frac{2 \sin^7(x)}{7} + \frac{\sin^5(x)}{5} + C$
15.  $-\frac{1}{6} \cos^5 x \sin x + \frac{1}{24} \cos^3 x \sin x + \frac{1}{16} \cos x \sin x + \frac{1}{16}x + C$

### Section 3.6: Trigonometric Substitution

1.  $x - \tan^{-1} x + C$
2.  $\frac{x}{\sqrt{1-x^2}} + C$
3.  $-\sqrt{1-x^2} + \sin^{-1} x + C$
4.  $\frac{x}{\sqrt{1+x^2}} + C$
5.  $\frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2} \sqrt{9-x^2} + C$
6.  $\ln |x + \sqrt{4+x^2}| + C$
7.  $\sqrt{16-x^2} \left( \frac{x^3}{4} - 2x \right) + 32 \sin^{-1} \left( \frac{x}{4} \right) + C$
8.  $-\frac{\sqrt{x^2+4}}{4x} + C$
9.  $\frac{x}{\sqrt{4x^2+1}}$
10.  $\frac{x-1}{\sqrt{2x-x^2}}$

### Section 3.7: Rational Functions

1.  $-x + \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$
2.  $9x - \frac{3}{2}x^2 + \frac{1}{3}x^3 - 27 \ln |3+x| + C$
3.  $x + \ln(1+x^2) + C$
4.  $\frac{1}{4} \ln \left| \frac{x-1}{x+3} \right| + C$
5.  $\frac{1}{10\sqrt{2}} \ln \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| - \frac{1}{5\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$
6.  $\frac{1}{x+1} + \frac{1}{2} \ln |x^2 - 1| + C$
7.  $-\frac{5x-6}{x^2-3x+2} + 4 \ln \left| \frac{x-1}{x-2} \right| + C$
8.  $\tan^{-1} x + \frac{5}{6} \ln \frac{x^2+1}{x^2+4} + C$
9.  $\frac{1}{2} \tan^{-1} x + \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} + C$
10.  $x^2 + 2 \ln |x+1| + 3 \ln |x-3| + C$
11.  $\tan^{-1} x - \frac{1}{x-1} + \ln \frac{x^2+1}{(x-1)^2} + C$
12.  $\frac{1}{2(x^2+1)} + \ln |x| - \frac{1}{2} \ln(x^2+1) + C$

$$13. \frac{9}{2} \ln(x-3) - 4 \ln(x-2) + \frac{1}{2} \ln(x-1) + C \quad 14. \frac{1}{4} \ln\left(\frac{x^2}{x^2-2x+2}\right) - \frac{1}{2} \tan^{-1}(1-x) + C$$

### Section 3.8: *t*-method

$$1. -\frac{\cos x}{2 \sin^2 x} + \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + C$$

$$2. \tan x - \sec x + C$$

$$3. \frac{1}{\cos x} + \frac{1}{3 \cos^3 x} + \ln \left| \tan \frac{x}{2} \right| + C$$

$$4. \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2 \tan(\frac{x}{2}) + 1}{\sqrt{3}} \right) + C$$

$$5. 2\sqrt{2} \tan^{-1} \left( \frac{\tan(\frac{x}{2})}{\sqrt{2}} \right) - x + C$$

$$6. \frac{1}{2}(x + \ln(\sin x + \cos x + 3)) - \frac{1}{\sqrt{7}} \tan^{-1} \left( \frac{2 \tan(\frac{x}{2}) + 1}{\sqrt{7}} \right) + C$$

### Section 3.9: Piecewise Functions

$$1. F(x) + C \text{ where } F(x) = \begin{cases} 2x^2 - x + 5, & x < 1 \\ 6\sqrt{x}, & x \geq 1 \end{cases}$$

$$2. F(x) + C \text{ where } F(x) = \begin{cases} \frac{e^{2x}}{2}, & x < 0 \\ \frac{2x + \sin 2x + 2}{4}, & x \geq 0 \end{cases}$$

$$3. \frac{|x-3|(x-3)}{2} + C$$

$$4. F(x) + C \text{ where } F(x) = \begin{cases} -\ln(1-x), & x < 0 \\ \ln(1+x), & x \geq 0 \end{cases}$$

$$5. F(x) + C \text{ where } F(x) = \begin{cases} x^3 - 12x, & x < -2 \\ -x^3 + 12x + 32, & -2 \leq x < 2 \\ x^3 - 12x + 64, & x \geq 2 \end{cases}$$

$$6. F(x) + C \text{ where } F(x) = \begin{cases} x(1 - \ln x), & 0 < x < 1 \\ x(\ln x - 1) + 2, & x \geq 1 \end{cases}$$