THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 6
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In addition to the following exercises on differentiation and curve sketching,
we will continue our discussion on Exercise 3 from the last tutorial.

Curve Sketching Techniques

Let $f : D \to \mathbb{R}$ be a twice differentiable function.
In order to sketch the graph of f, we should consider the following items.

1. Find the x, y- intercepts.
2. Find the relative extremum(s).
   If $f'(x_0) = 0$ and $x_0 \in D$, then x_0 is a critical point. Then apply second derivative test to check if they are the relative maximum or the relative minimum.

   $$x_0 \text{ is a relative maximum point if } f''(x_0) < 0;$$
   $$x_0 \text{ is a relative minimum point if } f''(x_0) > 0$$

   You can also test by finding $f'(x)$ for nearby $x > x_0$ and $x < x_0$.
3. Find the point(s) of inflexion.
   The point of inflexion, or inflection point is the point at which its sign of the second derivative changes from left to right. In other words, x_0 is the point of inflexion if either

   $$f''(x_0) = 0, \quad f''(x) > 0 \text{ for } x < x_0, \quad f''(x) < 0 \text{ for } x > x_0$$

   or

   $$f''(x_0) = 0, \quad f''(x) < 0 \text{ for } x < x_0, \quad f''(x) > 0 \text{ for } x > x_0$$
4. Find the interval(s) with concavity.
   The interval where f is concave upwards has $f''(x) > 0$ for x lying on the interval;
   The interval where f is concave downwards has $f''(x) < 0$ for x lying on the interval.
5. Find the equation(s) of asymptote.
   The equation of asymptote informs us how close the distance of the graph of f with the line given by the equation as x or y component tends to infinity.

   (i) Vertical asymptote
   $x = b$ is a vertical asymptote if either one of the following statements is true:

   $$\lim_{x \to b^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to b^+} f(x) = \pm\infty$$

   (ii) Horizontal asymptote
   $y = c$ is a horizontal asymptote if either one of the following statements is true:

   $$\lim_{x \to -\infty} f(x) = c \quad \text{or} \quad \lim_{x \to \infty} f(x) = c$$

   (ii) Oblique asymptote
   $y = mx + c$ is an oblique asymptote if either one of the following statements is true:

   $$\lim_{x \to -\infty} [f(x) - (mx + c)] = 0 \quad \text{or} \quad \lim_{x \to \infty} [f(x) - (mx + c)] = 0$$

Exercise 1:

For each of the function f,

$$\text{(i) } f(x) = \frac{x}{\sqrt{x^2+1}} \qquad\qquad \text{(ii) } f(x) = \frac{x^2}{x+1}$$

$$\text{(iii) } f(x) = (1+x)e^{-2x}$$

(1) Find, if any,

  (a) $f'(x)$ and $f''(x)$

  (b) The relative maximum and minimum

  (c) The point of inflexion

  (d) The interval where f is concave upwards / downwards

  (e) The equation of asymptote
  (For f in (iii), show that $e^x > 1 + x$ for $x > 0$)

(2) Sketch the graph of f.

Exercise 2:

Suppose $f : [-\pi, \pi] \to \mathbb{R}$ is the continuous function defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & x \in [-\pi, \pi] \setminus \{0\} \\ a & x = 0 \end{cases}$$

(a) Find the value of a.

(b) You are given that f is differentiable at $x = 0$, and $f'(0) = 0$.
Show that

  (i) $\tan x > x$ for any $x \in \left(0, \frac{\pi}{2}\right)$.

  (ii) $\tan x < x$ for any $x \in \left(\frac{\pi}{2}, \pi\right)$.

Hence show that there is only one critical point on $(-\pi, \pi)$.

(c) Find the absolute maximum and absolute minimum of f on $[-\pi, \pi]$.

Solution

Exercise 1:

(i) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

(a)

$$f'(x) = \frac{\sqrt{x^2+1} - \dfrac{x^2}{\sqrt{x^2+1}}}{x^2+1} = \frac{x^2+1-x^2}{(x^2+1)\sqrt{x^2+1}} = \frac{1}{(x^2+2)^{\frac{3}{2}}}$$

$$f''(x) = -\frac{3}{2}\frac{1}{(x^2+1)^{\frac{5}{2}}}2x = -\frac{3x}{(x^2+1)^{\frac{5}{2}}}$$

(b) By (a), $f'(x) > 0$ for all $x \in \mathbb{R}$. Hence, f is strictly increasing on $\mathbb{R}$. Therefore, there is no relative minimum and relative maximum points.

(c) Set $f''(x) = 0$. Then $x = 0$.

x	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	$+$	0	$-$

We have the inflexion point $(0, f(0)) = (0, 0)$.

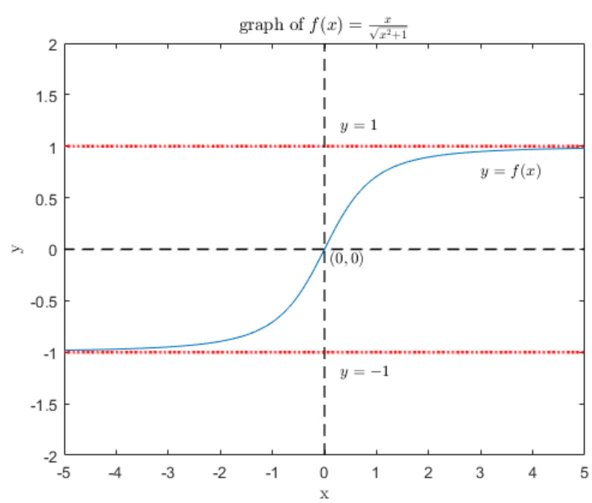
(d) By the table in (c), f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.

(e)

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} -\frac{1}{\sqrt{1+\frac{1}{x^2}}} = -1$$

The horizontal asymptotes are $y = -1$ and $y = 1$.



(ii) $f(x) = \dfrac{x^2}{1+x}$

(a)

$$f'(x) = \frac{x(x+2)}{(1+x)^2}$$

$$f''(x) = \frac{2}{(1+x)^3}$$

(b) Let $f'(x) = 0$. Then $x = 0$ or -2.
Note that $f''(0) = 2 > 0$ and $f''(-2) = -2 < 0$.
By the second derivative test,
$(0, 0)$ and $(-2, -4)$ is our minimum and maximum point respectively.

(c) There is no point of inflexion since $f''(x) \neq 0$.

(d)

x	$x < -1$	$x = -1$	$x > -1$
$f''(x)$	$-$	undefined	$+$

f is concave upward on $(-1, \infty)$ and concave downward on $(-\infty, -1)$.

(e)

$$\lim_{x \to -1^+} \frac{x^2}{1 + x} = +\infty$$

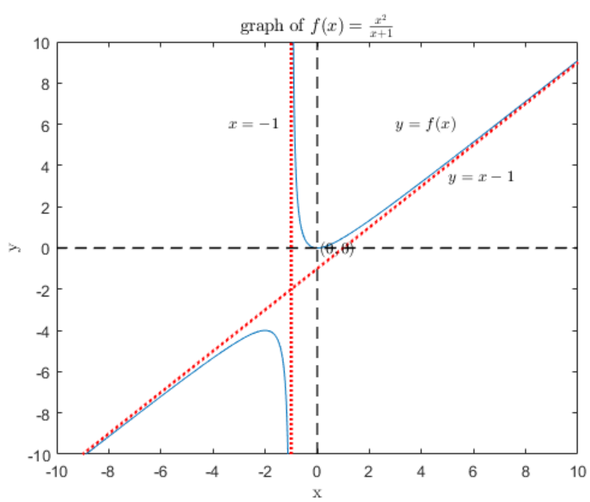
$$\lim_{x \to -1^-} \frac{x^2}{1 + x} = -\infty$$

The vertical asymptote is $x = -1$.
Let the oblique asymptote be $y = mx + b$

$$m = \lim_{x \to \infty} \frac{\frac{x^2}{1+x}}{x} = \lim_{x \to \infty} \frac{x}{1 + x} = 1$$

$$b = \lim_{x \to \infty} \frac{x^2}{1 + x} - x = \lim_{x \to \infty} -\frac{x}{1 + x} = -1$$

The oblique asymptote is $y = x - 1$.



(iii) $f(x) = (1 + x)e^{-2x}$

(a)

$$f'(x) = -(2x + 1)e^{-2x}$$

$$f''(x) = 4xe^{-2x}$$

(b) Let $f'(x) = 0$. Then $x = -\frac{1}{2}$.

Note that $f''\left(\frac{1}{2}\right) = -2e^{-1} < 0$.

By the second derivative test, $\left(-\frac{1}{2}, \frac{e}{2}\right)$ is our maximum point.

(c) Let $f''(x) = 0$. Then $x = 0$.

x	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	$-$	0	$+$

We have the inflexion point $(0, f(0)) = (0, 1)$.

(d) By the table in (c), f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$.

(e) By applying mean value theorem, for $x > 0$,

$$e^x > 1 + x \quad \text{(Verify it)}$$

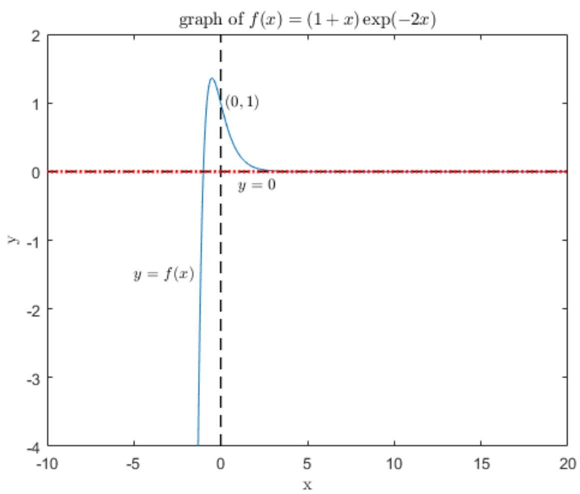
Hence

$$0 < (1 + x)e^{-2x} < e^{-x}$$

By squeeze theorem,

$$\lim_{x\to\infty} (1 + x)e^{-2x} = 0$$

The horizontal asymptote is $y = 0$.



Exercise 2:

(a) By continuity of f,

$$a = f(0) = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

(b) Let $g(u) = \tan u - u$. Let $x \in \left(0, \frac{\pi}{2}\right)$.

Note that $g'(u) = \sec^2 u - 1 > 0$ for $u \in \left(0, \frac{\pi}{2}\right)$.

Hence $\tan x - x = g(x) > g(0) = 0$.

Since $\tan x < 0$ for $x \in \left(\frac{\pi}{2}, \pi\right)$,

$g(x) = \tan x - x < 0$ for for $x \in \left(\frac{\pi}{2}, \pi\right)$.

Note that

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

For $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$,

$$f'(x) = \cos x \, \frac{x - \tan x}{x^2}$$

By the previous results, $\tan x \neq x$.

Observe that $f'(x)$ is an odd function on $(-\pi, \pi)$.
Hence $f'(x) \neq x$ for $x \in (-\pi, \pi) \setminus \left\{0, -\dfrac{\pi}{2}, \dfrac{\pi}{2}\right\}$.

Observe that $f'(0) = 0, \quad f'(\dfrac{\pi}{2}) = -\dfrac{4}{\pi^2}, \quad f'(-\dfrac{\pi}{2}) = \dfrac{4}{\pi^2}$.

Therefore, 0 is the only critical point in $(-\pi, \pi)$.

(c) By (b), $f'(x) < 0$ for $x \in \left(0, \dfrac{\pi}{2}\right)$.
Since f' is odd, $f'(x) > 0$ for $x \in \left(-\dfrac{\pi}{2}, 0\right)$.
Hence $(0, f(0)) = (0, 1)$ is a relative maximum point.

Note that
$$f(-\pi) = f(\pi) = 0$$
The absolute maximum attain at $x = 0$, and $f(x) = 1$.
The absolute minimum attain at $x = -\pi$ and $x = \pi$, and $f(x) = 0$.