THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 10
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Definite Integral

1. Theorem for definite integral

   Suppose f is continuous on $[a,b]$. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k$$

   Take $x_k = a + \frac{k}{n}(b-a)$ and $\Delta x_k = \frac{b-a}{n}$, we have

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + \frac{k}{n}(b-a)\right)\left(\frac{b-a}{n}\right)$$

2. Fundamental Theorem of Calculus

   (i) Suppose f is continuous on $[a,b]$. Define

$$F(x) = \int_a^x f(t)dt$$

   Then F is continuous on $[a,b]$, differentiable on (a,b) and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

   (ii) Suppose f is continuous on $[a,b]$. Define F as above. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

3. Corollary

   Suppose f is continuous on $[a,b]$. Let g,h be differentiable functions on $[a,b]$. Then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Exercise 1:

Evaluate the following limits.

(a) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+2k}$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5}$

(c) $\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{1}{\sqrt{2n^2+kn}}$

Exercise 2:

Evaluate the following integrals.

(a) $\int_0^2 |1-x|dx$

(b) $\int_0^{\frac{\pi}{2}} \sec^3 \frac{x}{2} \tan \frac{x}{2} dx$

Exercise 3:

Find $F'(x)$, where

(a) $F(x) = \int_0^{x^2} e^{t^2} dt$

(b) $F(x) = \int_0^{2x} \sin t \ln(1+t) dt$

(c) $F(x) = \int_{x^3}^{x^5} \ln t \cos e^t dt$

Exercise 4:

Define the function $f : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ by

$$f(x) = \int_1^x \cos(\sin t) dt$$

(a) Show that f is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

(b) Find all x satisfying $f(x) = 0$.

(c) Let g be the inverse of f. Find $g'(0)$.

Solution

Exercise 1:

(a) Let $f(x) = \dfrac{1}{1+2x}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+2k} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+2\cdot\frac{k}{n}}\frac{1}{n} = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$$

$$= \int_0^1 \frac{1}{1+2x}dx = \frac{1}{2}\Big[\ln|1+2x|\Big]_0^1 = \frac{\ln 3}{2}$$

(b) Let $f(x) = x^4$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^4}{n^5} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\frac{1}{n} = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$$

$$= \int_0^1 x^4 dx = \frac{1}{5}\Big[x^5\Big]_0^1 = \frac{1}{5}$$

(c) Let $f(x) = \dfrac{1}{\sqrt{2+x}}$.

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{\sqrt{2n^2+kn}} = \lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{\sqrt{2+\frac{k}{n}}}\frac{1}{n} = \int_0^2 f(x)dx$$

$$= \int_0^2 \frac{1}{\sqrt{2+x}}dx = \Big[2\sqrt{2+x}\Big]_0^2 = 4-2\sqrt{2}$$

Exercise 2:

(a)

$$\int_0^2 |1-x|dx = \int_0^1 |1-x|dx + \int_1^2 |1-x|dx = \int_0^1 (1-x)dx + \int_1^2 (x-1)dx$$
$$= \left[x-\frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^2 = 1$$

(b)

$$\int_0^{\frac{\pi}{2}} \sec^3\frac{x}{2}\tan\frac{x}{2}dx = 2\int_0^{\frac{\pi}{2}} \sec^3\frac{x}{2}\tan\frac{x}{2}d\frac{x}{2} = 2\int_0^{\frac{\pi}{2}} \sec^2\frac{x}{2}d\sec\frac{x}{2} = \frac{2}{3}\left[\sec^3\frac{x}{2}\right]_0^{\frac{\pi}{2}} = \frac{2\sqrt{8}-2}{3}$$

Exercise 3:

(a)

$$F'(x) = 2x\cdot e^{(x^2)^2} = 2xe^{x^4}$$

(b)

$$F'(x) = 2\cdot\sin(2x)\ln(1+2x) = 2\sin(2x)\ln(1+2x)$$

(c)

$$F'(x) = 5x^4\cdot\ln(x^5)\cos e^{x^5} - 3x^2\cdot\ln(x^3)\cos e^{x^3} = 5x^4\ln(x^5)\cos e^{x^5} - 3x^2\ln(x^3)\cos e^{x^3}$$

Exercise 4:

(a) By Fundamental Theorem of Calculus,

$$f'(x) = \cos(\sin x)$$

For $x \in (0, \frac{\pi}{2})$,

$$f'(x) = \cos(\sin x) > 0$$

Hence, f is strictly increasing on $(0, \frac{\pi}{2})$.

(b) Note that $f(1) = 0$ and f is strictly increasing. So the only x satisfying $f(x) = 0$ is $x = 1$.

(c) Note that

$$g(f(x)) = x$$
$$g'(f(x))f'(x) = 1$$

Since $f(1) = 0$, we have

$$g'(0)f'(1) = 1$$
$$g'(0) = \frac{1}{f'(1)}$$
$$g'(0) = \frac{1}{\cos \sin 1}$$