Tutorial 1

1. Let $(X, \mathcal{T})$ be a topo space. $A \subset X$

satisfies : for $\forall\ x \in A$
$\exists\ U$ open
s.t $x \in U \subset A$

Show that A is open

Pf : Above condition tells us that :

for $\forall\ x \in A$, one can find an open set U_x s.t $x \in U_x \subset A$

$$\Rightarrow \underbrace{\bigcup_{x \in A} \{x\}}_{= A} \subset \bigcup_{x \in A} U_x \subset A$$

$$\Rightarrow A = \bigcup_{x \in A} U_x \text{ is open}$$

Q2 Let X be a set.

$$\mathcal{T}_c = \left\{ U \subset X \;\middle|\; \begin{array}{l} U^c \text{ is countable} \\ \text{or } U^c = X \end{array} \right\}$$

Show that $(X, \mathcal{T}_c)$ is a topo space.

Pf: • $\emptyset^c = X \Rightarrow \emptyset \in \mathcal{T}_c$

$X^c = \emptyset$ countable $\Rightarrow X \in \mathcal{T}_c$

• We consider a collection of open sets $\{U_\alpha\}_{\alpha \in I}$

Need to prove $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.

<u>case 1</u> $U_\alpha = \emptyset$ for $\forall\ \alpha \in I$ ✓

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha = \emptyset \in \mathcal{T}$

<u>case 2</u> $U_{\alpha_0} \neq \emptyset$ for a $\alpha_0 \in I$

$\Rightarrow U_{\alpha_0}^c$ is countable

$\Rightarrow \left(\bigcup_{\alpha \in I} U_\alpha\right)^c = U_{\alpha_0}^c \cap \left(\bigcup_{\alpha \neq \alpha_0} U_\alpha\right)^c$ is countable

• Let $U_1, \dots, U_N$ are N open sets

Need to show $\bigcap_{i=1}^{N} U_i \in \mathcal{T}_c$

case 1 $U_i \neq \emptyset$ for all $i \in \{1, \dots, N\}$

i.e. U_i^c is countable for all $i \in \{1, \dots, N\}$

$\Rightarrow \left(\bigcap_{i=1}^{N} U_i\right)^c = \bigcup_{i=1}^{N} U_i^c$ is countable

Case 2 $\quad U_K = \phi$ for an integer $K \in \{1, \cdots, N\}$

$$\Longrightarrow \bigcap_{i=1}^{N} U_i = \phi \in \mathcal{T}$$

Q3. Consider the following topo on $\mathbb{R}^1$:

$\mathcal{T}_1$ = standard topo

$\mathcal{T}_2 = \mathcal{T}_c$ = co-finite topo

Show $\mathcal{T}_2 \subsetneq \mathcal{T}_1$

(b). Let $U \in \mathcal{T}_2$

If $U = \phi$ then $U \in \mathcal{T}_1$

If $U \neq \phi$ then U^c is finite

$$\begin{array}{c} U^c \\ \| \\ (x_1 < x_2 < \cdots < x_N)\ \{x_1, x_2, \dots, x_N\} \end{array}$$

$$\text{So } U = \underbrace{(-\infty, x_1)}_{\substack{\| \\ \bigcup_{i=1}^{+\infty} (x_1 - i,\, x_1)}} \cup (x_2, x_3) \cup \cdots \cup (x_{N-1}, x_N) \cdots \cup \underbrace{(x_N, +\infty)}_{\substack{\| \\ \bigcup_{i=1}^{+\infty} (x_N,\, x_N + i)}} \in \mathcal{T}_1$$

In all $U \in \mathcal{T}_1$ for $\forall\ U \in \mathcal{T}_3$

i.e. $\mathcal{T}_3 \subset \mathcal{T}_1$

On the other hand,

Note that $(0,1) \in \mathcal{T}_1$

but $(0,1) \notin \mathcal{T}_3$

Since $(0,1)^c$ is neither finite

nor $= \mathbb{R}$

So $\mathcal{T}_3 \subsetneq \mathcal{T}_1$

<u>Q4</u>. Show that

$$\mathcal{B}' \triangleq \{ (a,b) \mid \underset{a<b}{a, b} \in \mathbb{Q} \}$$

is a basis of $(\mathbb{R}, \mathcal{T}_{std})$

<u>Pf</u>: $\mathcal{B} = \{ (a,b) \mid a<b \}$ generates $\mathcal{T}_{std}$

$$\Downarrow$$

For each $x \in \mathbb{R}$ and $U \in \mathcal{T}_{std}$ that satisfying $x \in U$

$\exists$, $(a, b) \in \mathcal{B}$

s.t., $x \in (a, b) \subset U$

But one can always find $p, q \in \mathbb{Q}$

s.t $a < p < x < q < b$

i.e., $x \in \underset{\substack{\cap \\ \mathcal{B}'}}{(p, q)} \subseteq (a, b) \subset U$

$$\Downarrow$$

$\mathcal{B}'$ is a basis for $(\mathbb{R}, \mathcal{T}_{std})$