MATH 1030 Chapter 13

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

Reference.

Beezer, Ver 3.5 Section B (print version p233-238), Section D (print version p245-253)

Exercise.

- Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf (Replace $\mathbb{C}$ by $\mathbb{R}$)

  Section B p.88-92 C10, C11, C12, M20 Section D p.92-96 C21, C23, C30, C31, C35, C36, C37, M20, M21.

13.1 Basis

Definition 13.1. Let V be a vector space. Then a subset S of V is said to be a **basis** for V if

1. S is linearly independent.
2. $\operatorname{Span} S = V$, i.e. S spans V.

Remark. Most of the time V is a subspace of $\mathbb{R}^m$. Occasionally V is assumed to be a subspace of M_{mn} or P_n. It does not hurt to assume V is a subspace of $\mathbb{R}^m$.

Example 13.2. Let $V = \mathbb{R}^m$, then $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is a basis for V. (recall all the entries of $\mathbf{e}_i$ is zero, except the i-th entry being 1).

It is called the **standard basis**: Obviously B is linearly independent. Also, for any $\mathbf{v} \in V$, $\mathbf{v} = [\mathbf{v}]_1\mathbf{e}_1 + \cdots + [\mathbf{v}]_m\mathbf{e}_m \in \operatorname{Span} B$. So $\operatorname{Span} B = V$.

Example 13.3. Math major only

Consider $V = M_{22}$. Let:

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

Then $B = \{B_{11}, B_{12}, B_{21}, B_{22}\}$ is a basis for V.

Check: Obviously B is linearly independent (exercise). Also for any $A \in V$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aB_{11} + bB_{12} + cB_{21} + dB_{22}.$$

So $\operatorname{Span} B = M_{22}$.

Exercise 13.4. Math major only

Let $V = M_{mn}$.

For $1 \leq i \leq m$, $1 \leq j \leq n$, let B_{ij} be the $m \times n$ matrix with (i, j)-th entry equal to 1 and all other entries equal to 0.

Then $\{B_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for V.

Example 13.5. Math major only

Let $V = P_n$. Then $1, x, x^2, \ldots, x^n$ is a basis. It is easy to show that $S = \{1, x, x^2, \ldots, x^n\}$ is linearly independent. Also any polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is a linear combinations of S.

Example 13.6. A vector space can have different bases.

Consider the vector space $V = \mathbb{R}^2$.

Then,

$$S = \{\mathbf{e}_1, \mathbf{e}_2\}$$

is a basis for V, and:

$$S' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is also a basis.

13.2 Bases for Spans of Column vectors

13.2.1 Column Spaces and Systems of Equations

Definition 13.7 (Column Space of a Matrix)**.** Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n$. Then the **column space** of A, written $\mathcal{C}\left(A\right)$, is the subset of $\mathbb{R}^m$ consisting of all linear combinations of the columns of A,

$$\mathcal{C}\left(A\right) = \operatorname{Span}\left\{\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n\right\}$$

Theorem 13.8 (Column Spaces and Consistent Systems)**.** *Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{C}\left(A\right)$ if and only if $A\mathbf{x} = \mathbf{b}$ is consistent.*

Proof of Column Spaces and Consistent Systems. ($\Rightarrow$) Suppose $\mathbf{b} \in \mathcal{C}\left(A\right)$. Then we can write $\mathbf{b}$ as some linear combination of the columns of A. Then by Theorem 5.18 (Recognizing Consistency of a Linear System) we can use the scalars from this linear combination to form a solution to $A\mathbf{x} = \mathbf{b}$, so this system is consistent.

($\Leftarrow$) If $A\mathbf{x} = \mathbf{b}$ is consistent, there is a solution that may be used with Theorem 5.18 (Recognizing Consistency of a Linear System) to write $\mathbf{b}$ as a linear combination of the columns of A. This qualifies $\mathbf{b}$ for membership in $\mathcal{C}\left(A\right)$. ☐

This theorem tells us that asking if the vector equation $A\mathbf{x} = \mathbf{b}$ has a solution is exactly the same question as asking if $\mathbf{b}$ is in the column space of A.

Thus, an alternative (and popular) definition of the column space of an $m \times n$ matrix A is

$$\mathcal{C}\left(A\right) = \left\{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\right\} = \left\{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\right\} \subseteq \mathbb{R}^m$$

Example 13.9. Consider the column space of the 3×4 matrix A,

$$A = \begin{bmatrix} 3 & 2 & 1 & -4 \\ -1 & 1 & -2 & 3 \\ 2 & -4 & 6 & -8 \end{bmatrix}$$

Show that $\mathbf{v} = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix}$ is in the column space of A, $\mathbf{v} \in \mathcal{C}\left(A\right)$. The above theorem says that we need to check the consistency of $\mathcal{LS}(A, v)$. From the augmented matrix and row-reduce,

$$\begin{bmatrix} 3 & 2 & 1 & -4 & 18 \\ -1 & 1 & -2 & 3 & -6 \\ 2 & -4 & 6 & -8 & 12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -2 & 6 \\ 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column is not a pivot column, so the system is consistent and hence $v \in \mathcal{C}(A)$. In fact, we have

$$\mathbf{v} = 6\mathbf{A}_1.$$

Next we show that $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ is not in the column space of A, $\mathbf{w} \notin \mathcal{C}(A)$. The above theorem says that we need to check the consistency of $\mathcal{LS}(A, w)$. From the augmented matrix and row-reduce,

$$\begin{bmatrix} 3 & 2 & 1 & -4 & 2 \\ -1 & 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & -8 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -2 & 0 \\ 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since the final column is a pivot column, the system is inconsistent and therefore $\mathbf{w} \notin \mathcal{C}(A)$.

13.2.2 Column Space Spanned by Original Columns

Theorem 13.10 (Basis of the Column Space)**.** *Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of indices for the pivot columns of B. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then:*

1. *T is a linearly independent set.*
2. *$\mathcal{C}(A) = \operatorname{Span} T$.*

Hence, T is a basis of $\mathcal{C}(A)$.

Lemma 13.11. *If a matrix A is row-equivalent to B, then any matrix formed with a subset of columns of A is row-equivalent to the matrix formed with the corresponding subset of columns (i.e. with the same indices and conforming to the same order) of B.*

Proof of Lemma 13.11. **Exercise**. □

Proof of Basis of the Column Space. Let:

$$A' = [\mathbf{A}_{d_1}|\mathbf{A}_{d_2}|\cdots|\mathbf{A}_{d_r}],$$

$$B' = [\mathbf{B}_{d_1}|\mathbf{B}_{d_2}|\cdots|\mathbf{B}_{d_r}].$$

Then, A' and B' are row-equivalent by the previous lemma.

1. Since $d_1, d_2, \cdots d_r$ are the indices of the pivot columns of B, which is in RREF, we have:
$$B' = \left[\begin{array}{c} I_r \\ \hline \mathcal{O}_{(m-r)\times r} \end{array}\right].$$
(If $r = m$, then $B' = I_r$.)

   Hence, $B'\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$.

   Since A' is row-equivalent to B', we have $A'\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$, which implies that the columns of A', i.e. the elements of T, are linearly independent.

2. Given any integer $1 \leq j \leq n$ which does not lie in D (i.e. the index of a non-pivot column of B), consider the linear system $\mathcal{LS}(A', \mathbf{A}_j)$ (or equivalently $A'\mathbf{x} = \mathbf{A}_j$).

   By the previous lemma, the corresponding the augmented matrix $[A'|\mathbf{A}_j]$ is row-equivalent to $[B'|\mathbf{B}_j]$, which has the form:
$$\left[\begin{array}{c|c} & * \\ & * \\ I_r & \vdots \\ & * \\ \hline & 0 \\ & 0 \\ \mathcal{O}_{(m-r)\times r} & \vdots \\ & 0 \end{array}\right].$$
Hence, the linear system $\mathcal{LS}(A', \mathbf{A}_j)$ is consistent (in fact, with one unique solution), which implies that $\mathbf{A}_j$ lies in the span of $\{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \ldots, \mathbf{A}_{d_r}\}$.

   This holds for all $j \notin D$. Hence, by Theorem 11.11, we have:
$$\mathcal{C}(A) = \operatorname{Span}\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\} = \operatorname{Span}\{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \ldots, \mathbf{A}_{d_r}\} = \operatorname{Span} T.$$

□

Example 13.12. Consider the 5×7 matrix A,
$$\begin{bmatrix} 2 & 4 & 1 & -1 & 1 & 4 & 4 \\ 1 & 2 & 1 & 0 & 2 & 4 & 7 \\ 0 & 0 & 1 & 4 & 1 & 8 & 7 \\ 1 & 2 & -1 & 2 & 1 & 9 & 6 \\ -2 & -4 & 1 & 3 & -1 & -2 & -2 \end{bmatrix}$$

The column space of A is

$$\mathcal{C}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 8 \\ 9 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 7 \\ 6 \\ -2 \end{bmatrix} \right\}$$

While this is a concise description of an infinite set, we might be able to describe the span with fewer than seven vectors. Now we row-reduce,

$$\begin{bmatrix} 2 & 4 & 1 & -1 & 1 & 4 & 4 \\ 1 & 2 & 1 & 0 & 2 & 4 & 7 \\ 0 & 0 & 1 & 4 & 1 & 8 & 7 \\ 1 & 2 & -1 & 2 & 1 & 9 & 6 \\ -2 & -4 & 1 & 3 & -1 & -2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are $D = \{1,\ 3,\ 4,\ 5\}$, so we can create the set

$$T = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and know that $\mathcal{C}(A) = \text{Span}\, T$ and T is a linearly independent set of columns from the set of columns of A.

Hence, T is a basis of $\mathcal{C}(A)$.

Example 13.13. Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix},$$

find $\mathcal{C}(A)$.

$$A \xrightarrow{\text{RREF}} B = \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The indices of the pivot columns are $D = \{1, 3, 4\}$. Hence $\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\}$ is a basis of $\mathcal{C}(A)$.

13.3 Bases and Nonsingular Matrices

Theorem 13.14 (Column Space of a Nonsingular Matrix)**.** *Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{R}^n$.*

Proof of Column Space of a Nonsingular Matrix. See Theorem 11.9. □

Hence, we may rephrase Theorem 12.17 (Nonsingular Matrix Equivalences, Round 2) as follows:

Theorem 13.15 (Nonsingular Matrix Equivalences)**.** *Suppose that A is an $m \times m$ square matrix. The following are equivalent:*

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *A is invertible.*
6. *The columns of A form a linearly independent set.*
7. *The columns space $\mathcal{C}(A)$ is equal to $\mathbb{R}^m$.*
8. *The columns of A form a basis for $\mathbb{R}^m$.*

Example 13.16. Consider $S' = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Let

$$A = [\mathbf{v}_1 | \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise. The matrix A is nonsingular.

Hence, S' is a basis for $\mathbb{R}^2$.

Example 13.17.

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}.$$

It may be shown that A is row equivalent to the 3×3 identity matrix.

Hence A is nonsingular, so the columns of A form a basis for $\mathbb{R}^3$.

Example 13.18. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

We can show that A is nonsingular as $A \xrightarrow{\text{RREF}} I_4$. So $\mathcal{C}(A) = \mathbb{R}^4$.

13.4 Row Space of a Matrix

Definition 13.19 (Row Space of a Matrix)**.** Suppose A is an $m \times n$ matrix. The **row space** of A, $\mathcal{R}(A)$ is column space $\mathcal{C}(A^t)$ of A^t.

Informally, the row space is the set of all linear combinations of the rows of A. However, we write the rows as column vectors, thus the necessity of using the transpose to make the rows into columns. Additionally, with the row space defined in terms of the column space, all of the previous results of this section can be applied to row spaces.

Notice that if A is a rectangular $m \times n$ matrix, then $\mathcal{C}(A) \subseteq \mathbb{R}^m$, while $\mathcal{R}(A) \subseteq \mathbb{R}^n$ and the two sets are not comparable since they do not even hold objects of the same type. However, when A is square of size n, both $\mathcal{C}(A)$ and $\mathcal{R}(A)$ are subsets of $\mathbb{R}^n$, though usually the sets will not be equal.

Example 13.20. Find $\mathcal{R}(A)$ for

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}.$$

To build the row space, we transpose the matrix,

$$A^t = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 4 & 8 & 0 & -4 \\ 0 & -1 & 2 & 2 \\ -1 & 3 & -3 & 4 \\ 0 & 9 & -4 & 8 \\ 7 & -13 & 12 & -31 \\ -9 & 7 & -8 & 37 \end{bmatrix}$$

Then the columns of this matrix are used in a span to build the row space,

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) = \text{Span}\left\{\begin{bmatrix}1\\4\\0\\-1\\0\\7\\-9\end{bmatrix}, \begin{bmatrix}2\\8\\-1\\3\\9\\-13\\7\end{bmatrix}, \begin{bmatrix}0\\0\\2\\-3\\-4\\12\\-8\end{bmatrix}, \begin{bmatrix}-1\\-4\\2\\4\\8\\-31\\37\end{bmatrix}\right\}.$$

First, row-reduce A^t,

$$\begin{bmatrix}\boxed{1} & 0 & 0 & -\frac{31}{7}\\ 0 & \boxed{1} & 0 & \frac{12}{7}\\ 0 & 0 & \boxed{1} & \frac{13}{7}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Since the pivot columns have indices $D = \{1,\, 2,\, 3\}$, the column space of A^t can be spanned by just the first three columns of A^t,

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) = \text{Span}\left\{\begin{bmatrix}1\\4\\0\\-1\\0\\7\\-9\end{bmatrix}, \begin{bmatrix}2\\8\\-1\\3\\9\\-13\\7\end{bmatrix}, \begin{bmatrix}0\\0\\2\\-3\\-4\\12\\-8\end{bmatrix}\right\}.$$

Theorem 13.21 (Row-Equivalent Matrices have Equal Row Spaces)**.** *Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.*

Proof of Row-Equivalent Matrices have Equal Row Spaces. Observe that if B is obtained from A via a row operation of the type $R_i \leftrightarrow R_j$, then the rows of B are the same as the rows of A, and hence the columns of B^t are still the same as the columns of A^t, only with the order changed. Hence,

$$\mathcal{R}(B) = \mathcal{C}\left(B^t\right) = \mathcal{C}\left(A^t\right) = \mathcal{R}(A).$$

If B is obtained from A via a row operation of the type αR_i ($\alpha \neq 0$), then the i-th column of B^t is equal to α times the i-th column of A^t, and the other columns remain the same as those of A^t with the corresponding indices.

In paricular, the i-th column of B^t is a linear combination of the columns of A^t.

Hence, the columns of B^t all lie in $\mathcal{C}(A^t)$, which in turn implies that:

$$\mathcal{R}(B) = \mathcal{C}\left(B^t\right) \subseteq \mathcal{C}\left(A^t\right) = \mathcal{R}(A).$$

On the other hand, if B is obtained from A via αR_i, then A is obtained from B via $\left(\frac{1}{\alpha}\right) R_i$. So, by the same argument as before we have:

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) \subseteq \mathcal{C}\left(B^t\right) = \mathcal{R}(B).$$

Hence, $\mathcal{R}(B) = \mathcal{R}(A)$.

If B is obtained from A via a row operation of the type $\alpha R_i + R_j$, then:

$$\left[B^t\right]_j = \alpha \left[A^t\right]_i + \left[A^t\right]_j,$$

and the other columns of B^t remain the same as those of A^t with the corresponding indices.

In paricular, the i-th column of B^t is a linear combination of the columns of A^t.

Hence, the columns of B^t all lie in $\mathcal{C}(A^t)$, which in turn implies that:

$$\mathcal{R}(B) = \mathcal{C}\left(B^t\right) \subseteq \mathcal{C}\left(A^t\right) = \mathcal{R}(A).$$

On the other hand, if B is obtained from A via $\alpha R_i + R_j$, then A is obtained from B via $(-\alpha)R_i + R_j$. So, by the same argument as before we have:

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) \subseteq \mathcal{C}(B^t) = \mathcal{R}(B).$$

Hence, $\mathcal{R}(B) = \mathcal{R}(A)$.

We now see that the row space of a matrix remains unchanged after any application of a row operation.

Hence, $\mathcal{R}(B) = \mathcal{R}(A)$ if B is row-equivalent to A, since by the definition of row-equivalence (Definition 4.15 (Row-Equivalent Matrices)) B is obtained by A via a series of row operations. $\square$

Example 13.22. Row spaces of two row-equivalent matrices

The matrices

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

are row-equivalent via a sequence of two row operations.

Hence by the above theorem

$$\mathcal{R}(A) = \text{Span}\left\{\begin{bmatrix}2\\-1\\3\\4\end{bmatrix}, \begin{bmatrix}5\\2\\-2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\0\\6\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\1\\0\\6\end{bmatrix}, \begin{bmatrix}3\\0\\-2\\-9\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\4\end{bmatrix}\right\} = \mathcal{R}(B)$$

Theorem 13.23 (Basis for the Row Space)**.** *Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then*

1. $\mathcal{R}(A) = \text{Span}\, S$.
2. *S is a linearly independent set.*

Proof of Basis for the Row Space. From Theorem Theorem 13.21 (Row-Equivalent Matrices have Equal Row Spaces). we know that $\mathcal{R}(A) = \mathcal{R}(B)$. If B has any zero rows, these are columns of B^t that are the zero vector. We can safely toss out the zero vector in the span construction, since it can be recreated from the nonzero vectors by a linear combination where all the scalars are zero. So $\mathcal{R}(A) = \text{Span}\, S$.

Suppose B has r nonzero rows and let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ denote the indices of the pivot columns of B. Denote the r column vectors of B^t, the vectors in S, as $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_r$. To show that S is linearly independent, start with a relation of linear dependence

$$\alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \alpha_3\mathbf{B}_3 + \cdots + \alpha_r\mathbf{B}_r = \mathbf{0}$$

Now consider this vector equality in location d_i. Since B is in reduced row-echelon form, the entries of column d_i of B are all zero, except for a leading 1 in row i. Thus, in B^t, row d_i is all zeros, excepting a 1 in column i. So, for $1 \le i \le r$,

$$\begin{aligned}
0 &= [\mathbf{0}]_{d_i} \\
&= [\alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \alpha_3\mathbf{B}_3 + \cdots + \alpha_r\mathbf{B}_r]_{d_i} \\
&= [\alpha_1\mathbf{B}_1]_{d_i} + [\alpha_2\mathbf{B}_2]_{d_i} + [\alpha_3\mathbf{B}_3]_{d_i} + \cdots + [\alpha_r\mathbf{B}_r]_{d_i} \\
&= \alpha_1 [\mathbf{B}_1]_{d_i} + \alpha_2 [\mathbf{B}_2]_{d_i} + \alpha_3 [\mathbf{B}_3]_{d_i} + \cdots + \alpha_r [\mathbf{B}_r]_{d_i} \\
&= \alpha_1(0) + \alpha_2(0) + \alpha_3(0) + \cdots + \alpha_i(1) + \cdots + \alpha_r(0) \\
&= \alpha_i
\end{aligned}$$

So we conclude that $\alpha_i = 0$ for all $1 \le i \le r$, establishing the linear independence of S. □

Theorem 13.24 (Column Space Row Space Transpose)**.** *Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.*

Proof of Column Space, Row Space, Transpose.

$$\mathcal{C}(A) = \mathcal{C}\left(\left(A^t\right)^t\right) = \mathcal{R}\left(A^t\right)$$

□

Example 13.25. Column space from row operations

Let

$$S = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 8 \\ 0 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 9 \\ -4 \\ 8 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 7 \\ -13 \\ 12 \\ -31 \end{bmatrix}, \mathbf{v}_7 = \begin{bmatrix} -9 \\ 7 \\ -8 \\ 37 \end{bmatrix}\right.$$

Find a basis for $\operatorname{Span} S$.

$$A = [\mathbf{v}_1|\cdots|\mathbf{v}_7] = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}.$$

Method 1

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$T = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} \right\}.$$

Then T is a basis for $\operatorname{Span} S = \mathcal{C}(A)$.

Method 2 The transpose of A is

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 4 & 8 & 0 & -4 \\ 0 & -1 & 2 & 2 \\ -1 & 3 & -3 & 4 \\ 0 & 9 & -4 & 8 \\ 7 & -13 & 12 & -31 \\ -9 & 7 & -8 & 37 \end{bmatrix}.$$

Row-reduced this becomes,

$$D = \begin{bmatrix} \boxed{1} & 0 & 0 & -\frac{31}{7} \\ 0 & \boxed{1} & 0 & \frac{12}{7} \\ 0 & 0 & \boxed{1} & \frac{13}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can take

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix} \right\}.$$

T is a basis for $\mathcal{C}(A) = \operatorname{Span} S$.

Remark. Both methods describe algorithms to find bases (i.e., linear independent set the generate the column space) for the column space. Here are the differences.

1. In method 1, we find a subset of columns that forms a basis. However in method 2, the basis is not a subset of columns.

2. Given a vector $\mathbf{b} \in \mathcal{C}(A)$, it is easier to express it as a linear combination of the basis given by method 2.

Theorem 13.26. *Let S be a finite subset of $\mathbb{R}^m$. Then, a basis for* $\operatorname{Span} S$ *exists.*

In fact, there exists a subset T of S such that T is a basis for $\operatorname{Span} S$ *(see Theorem 13.10 (Basis of the Column Space)).*

13.5 Bases of Null Spaces

In this section, we will find a linearly independent set that spans a null space. Recall that, by Theorem 11.12, there exists a particular set of $n - r$ vectors that could be used to span the null space of a matrix.

Example 13.27. Linear independence of null space basis

Suppose that we are interested in the null space of a 3×7 matrix A which row-reduces to

$$B = \begin{bmatrix} \boxed{1} & 0 & -2 & 4 & 0 & 3 & 9 \\ 0 & \boxed{1} & 5 & 6 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 8 & -5 \end{bmatrix}.$$

Then $F = \{3,\, 4,\, 6,\, 7\}$ is the set of indices for our four free variables that would be used in a description of the solution set for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Applying Theorem 7.6, we can begin to construct a set of four vectors whose span is the null space of A, a set of vectors we will refer to as T.

$$\begin{aligned}\mathcal{N}(A) &= \operatorname{Span} T = \operatorname{Span}\{\mathbf{z}_1,\, \mathbf{z}_2,\, \mathbf{z}_3,\, \mathbf{z}_4\} \\ &= \operatorname{Span}\left\{\begin{bmatrix} \\ \\ 1 \\ 0 \\ \\ 0 \\ 0\end{bmatrix},\, \begin{bmatrix} \\ \\ 0 \\ 1 \\ \\ 0 \\ 0\end{bmatrix},\, \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 1 \\ 0\end{bmatrix},\, \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 0 \\ 1\end{bmatrix}\right\}\end{aligned}$$

So far, we have constructed as much of these individual vectors as we can, based just on the knowledge of the contents of the set F. This has allowed us to determine the entries in slots 3, 4, 6 and 7, while we have left slots 1, 2 and 5 blank. Without doing any more, let us ask if T is linearly independent? Begin with a relation of linear dependence on T, and see what we can learn about the scalars.

$$\begin{aligned}\mathbf{0} &= \alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \alpha_3\mathbf{z}_3 + \alpha_4\mathbf{z}_4 \\ \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix} &= \alpha_1\begin{bmatrix} \\ \\ 1 \\ 0 \\ \\ 0 \\ 0\end{bmatrix} + \alpha_2\begin{bmatrix} \\ \\ 0 \\ 1 \\ \\ 0 \\ 0\end{bmatrix} + \alpha_3\begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 1 \\ 0\end{bmatrix} + \alpha_4\begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 0 \\ 1\end{bmatrix} \\ &= \begin{bmatrix} \\ \\ \alpha_1 \\ 0 \\ \\ 0 \\ 0\end{bmatrix} + \begin{bmatrix} \\ \\ 0 \\ \alpha_2 \\ \\ 0 \\ 0\end{bmatrix} + \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ \alpha_3 \\ 0\end{bmatrix} + \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 0 \\ \alpha_4\end{bmatrix} = \begin{bmatrix} \\ \\ \alpha_1 \\ \alpha_2 \\ \\ \alpha_3 \\ \alpha_4\end{bmatrix}\end{aligned}$$

Applying the equalities of vectors, we see that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. So the only relation of linear dependence on the set T is the trivial one. By the definition of linear independence, the set T is linearly independent. The important feature of this example is how the **pattern of zeros and ones** in the four vectors led to the conclusion of linear independence.

Theorem 13.28 (Basis for Null Spaces)**.** *Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices of B which are and are not, respectively, pivot columns. Construct the $n - r$ vectors $\mathbf{z}_j$, $1 \leq j \leq n - r$ of size n as*

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}$$

(In fact $\mathbf{z}_j$ corresponding to the solution $x_{f_j} = 1$ and $x_{f_k} = 0$ for $k \neq j$.) Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \operatorname{Span} S$.
2. *S is a linearly independent set.*

Proof of Basis for Null Spaces. Study the above example. You can skip the proof for now. Notice first that the vectors $\mathbf{z}_j$, $1 \leq j \leq n - r$, are the same as the $n - r$ vectors defined in Theorem 11.12. Also, the hypotheses of Theorem 11.12 are the same as the hypotheses of the theorem we are currently proving. So Theorem 11.12 tells us that $\mathcal{N}(A) = \operatorname{Span} S$. That was the easy half, but the second part is not much harder. What is new here is the claim that S is a linearly independent set.

To prove the linear independence of a set, we need to start with a relation of linear dependence and somehow conclude that the scalars involved must all be zero, i.e., that the relation of linear dependence is trivial. So, we start with and equation of the form

$$\alpha_1 \mathbf{z}_1 + \alpha_2 \mathbf{z}_2 + \alpha_3 \mathbf{z}_3 + \cdots + \alpha_{n-r} \mathbf{z}_{n-r} = \mathbf{0}.$$

For each j, $1 \leq j \leq n - r$, consider the equality of the individual entries of the vectors on both sides of this equality in position f_j:

$$\begin{aligned}
0 &= [\mathbf{0}]_{f_j} \\
&= [\alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \alpha_3\mathbf{z}_3 + \cdots + \alpha_{n-r}\mathbf{z}_{n-r}]_{f_j} \\
&= [\alpha_1\mathbf{z}_1]_{f_j} + [\alpha_2\mathbf{z}_2]_{f_j} + [\alpha_3\mathbf{z}_3]_{f_j} + \cdots + [\alpha_{n-r}\mathbf{z}_{n-r}]_{f_j} \\
&= \alpha_1 [\mathbf{z}_1]_{f_j} + \alpha_2 [\mathbf{z}_2]_{f_j} + \alpha_3 [\mathbf{z}_3]_{f_j} + \cdots + \\
&\qquad \alpha_{j-1} [\mathbf{z}_{j-1}]_{f_j} + \alpha_j [\mathbf{z}_j]_{f_j} + \alpha_{j+1} [\mathbf{z}_{j+1}]_{f_j} + \cdots + \\
&\qquad \alpha_{n-r} [\mathbf{z}_{n-r}]_{f_j} \\
&= \alpha_1(0) + \alpha_2(0) + \alpha_3(0) + \cdots + \\
&\qquad \alpha_{j-1}(0) + \alpha_j(1) + \alpha_{j+1}(0) + \cdots + \alpha_{n-r}(0) && \text{definition of } \mathbf{z}_j \\
&= \alpha_j
\end{aligned}$$

So for all j, $1 \leq j \leq n-r$, we have $\alpha_j = 0$. Hence, the only relation of linear dependence on $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\}$ is the trivial one. By the definition of linear independence, the set is linearly independent, as desired. □

Example 13.29. Find the null space of the matrix

$$A = \begin{bmatrix} -2 & -1 & -2 & -4 & 4 \\ -6 & -5 & -4 & -4 & 6 \\ 10 & 7 & 7 & 10 & -13 \\ -7 & -5 & -6 & -9 & 10 \\ -4 & -3 & -4 & -6 & 6 \end{bmatrix}.$$

Solution. The RREF of A is:

$$B = \begin{bmatrix} \boxed{1} & 0 & 0 & 1 & -2 \\ 0 & \boxed{1} & 0 & -2 & 2 \\ 0 & 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The free variables are x_4 and x_5.

Setting $x_4 = 1$ and $x_5 = 0$ gives:

$$\mathbf{z}_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Setting instead $x_4 = 0$ and $x_5 = 1$ gives

$$\mathbf{z}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\mathcal{N}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note that the spanning set on the right is linearly independent. So, it forms a basis for $\mathcal{N}(A)$.