THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 17 Real Integrals

17.1 Trigonometric Functions over a Range of 2π

An important application of complex contour integration is to evaluate certain real definite integrals. There is not much theory behind so we mostly illustrate the key points by examples.

Suppose we are to find $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1+a\sin\theta}$ where $0<a<1$.

There are two crucial things in this type of integrals.

- The lower limit and upper limit of the integral are 0 and 2π. It is a range of 2π, i.e., over the interval $[\alpha, \alpha+2\pi]$. That is the parameter range of the unit circle, $\mathbb{S}^1$, with center 0 and radius 1.
- The integrand only involves trigonometric functions, or essentially made up of sine and consine.

To summarize, this method works for the following type of integral (or something that can be transformed to such form)

$$\int_\alpha^{\alpha+2\pi} F(\cos\theta, \sin\theta)\, d\theta\,.$$

Note that $\sin\theta = \dfrac{1}{2\mathbf{i}}\left(e^{\mathbf{i}\theta} - e^{-\mathbf{i}\theta}\right)$ and $\cos\theta = \dfrac{1}{2}\left(e^{\mathbf{i}\theta}+e^{-\mathbf{i}\theta}\right)$. In addition, on the unit circle $\mathbb{S}^1$, we have $z = e^{\mathbf{i}\theta}$ and $1/z = e^{-\mathbf{i}\theta}$. Thus,

$$\frac{1}{1+a\sin\theta} = \frac{1}{1+\frac{a}{2\mathbf{i}}\left(z-\frac{1}{z}\right)}\,.$$

Also, if $z = e^{\mathbf{i}\theta}$, then $dz = \mathbf{i}e^{\mathbf{i}\theta}\, d\theta = \mathbf{i}z\, d\theta$. Therefore, we have

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1+a\sin\theta} = \int_{\mathbb{S}^1} \frac{1}{1+\frac{a}{2\mathbf{i}}\left(z-\frac{1}{z}\right)}\,\frac{1}{\mathbf{i}z}\, dz = \int_{\mathbb{S}^1} \frac{2}{a\left(z^2+\frac{2\mathbf{i}}{a}z-1\right)}\, dz\,.$$

Note that $f(z) = \dfrac{2/a}{z^2+\frac{2\mathbf{i}}{a}z-1} = \dfrac{2/a}{(z-z_1)(z-z_2)}$ where

$$z_1 = \frac{-\mathbf{i}+\mathbf{i}\sqrt{1-a^2}}{a}, \qquad z_1 = \frac{-\mathbf{i}-\mathbf{i}\sqrt{1-a^2}}{a}.$$

For $0<a<1$, it is easy to verify that $|z_1|<1$ and $|z_2|>1$. Thus,

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{1+a\sin\theta} &= \int_{\mathbb{S}^1} \frac{2}{a\left(z^2+\frac{2\mathbf{i}}{a}z-1\right)}\, dz = 2\pi\mathbf{i}\,\mathrm{Res}(f, z_1)\\ &= 2\pi\mathbf{i}\,\frac{2/a}{z_1-z_2} = \frac{2\pi\mathbf{i}\cdot(2/a)}{\left(2\mathbf{i}\sqrt{1-a^2}\right)/a} = \frac{2\pi}{\sqrt{1-a^2}}\,.\end{aligned}$$

Note that if $a\geq 1$, the original real integral becomes an improper integral. The expressions for the two zeros z_1, z_2 are different and they lie on $\mathbb{S}^1$.

17.2 Improper Integrals

We assume that the reader is familiar with the knowledge of the definite integral of a function on a closed integral $[a, b]$, i.e.,

$$\int_a^b f(x)\,dx\,.$$

Let us first recall the meaning of improper integrals. There are two possibilities that lead to an improper integral, the integrand function is undefined or the domain is not a closed interval. Nevertheless, we can summarize them into the following.

Definition 17.1. Let f be a continuous function on the interval $(a, b]$ or $[b, \infty)$ for $a, b \in \mathbb{R}$. Then, provided that the limits exist,

$$\int_a^b f(x)\,dx :\stackrel{\text{def}}{=\!=} \lim_{\delta\to 0}\int_{a+\delta}^b f(x)\,dx\,,$$
$$\int_b^\infty f(x)\,dx :\stackrel{\text{def}}{=\!=} \lim_{R\to\infty}\int_b^R f(x)\,dx\,.$$

With such a definition, all the other improper integrals can be reduced to these two situations. For example,

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{R_1\to\infty}\int_{-R_1}^0 f(x)\,dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)\,dx\,.$$

Note that in the above, the two limits $R_1 \to \infty$ and $R_2 \to \infty$ are independent. There is a concept called the *principal value* of an improper integral, in which the limits are related,

$$\text{P.V.}\int_{-\infty}^\infty f(x)\,dx :\stackrel{\text{def}}{=\!=} \lim_{R\to\infty}\int_{-R}^R f(x)\,dx\,.$$

The principal value and the true improper integral coincide if the integrand is an even function.

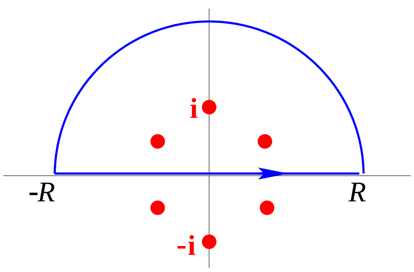
Example 17.2. Consider $f(x) = x$ on $(-\infty, \infty)$. We have

$$\text{P.V.}\int_{-\infty}^\infty f(x)\,dx = 0, \qquad \text{while} \qquad \int_{-\infty}^\infty f(x)\,dx \text{ diverges}\,.$$

17.2.1 Improper Integral of an Even Function

Example 17.3. Let us consider the example $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\,\text{P.V.}\int_{-\infty}^\infty \frac{x^2\,dx}{x^6+1}$.

Note that the two equalities above are true because $f(x) = \dfrac{x^2}{x^6+1}$ is an even function. Take the contour $\Gamma = (\gamma_1, \gamma_2)$ as in the picture, in which γ_1 is along the real axis and γ_2 is a semi-circle and the red dots represent the singularities.



Mathematically, on γ_1, $z = t$ for $t \in [-R, R]$; on γ_2, $z = Re^{\mathbf{i}t}$, $t \in [0, \pi]$. Let

$$f(z) = \frac{z^2}{z^6+1} = \frac{z^2}{(z-z_0)(z-\omega z_0)\cdots(z-\omega^5 z_0)}, \qquad \text{where } z_0 = e^{\mathbf{i}\pi/6},\ \omega = e^{2\mathbf{i}\pi/6}.$$

It can be seen that $z_0, \omega z_0, \omega^2 z_0$ lie inside Γ while $\omega^3 z_0, \omega^4 z_0, \omega^5 z_0$ are outside. Therefore,

$$\begin{aligned}\int_\Gamma f(z)\,dz &= 2\pi\mathbf{i}\left[\operatorname{Res}(f, z_0) + \operatorname{Res}(f, \omega z_0) + \operatorname{Res}(f, \omega^2 z_0)\right] \\ &= 2\pi\mathbf{i}\left(\frac{1}{6\mathbf{i}} - \frac{1}{6\mathbf{i}} + \frac{1}{6\mathbf{i}}\right) = \frac{\pi}{3}.\end{aligned}$$

Note that in the above, it is easy to consider $h(z) = z^2$ and $q(z) = z^6 + 1$ and get the residues by $h(z)/q'(z)$. Therefore, for any $R > 1$, we have

$$\frac{\pi}{3} = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz. \tag{$*$}$$

On the other hand, along the contour γ_1,

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} f(x)\,dx = \int_{-R}^{R} \frac{x^2}{x^6+1}\,dx \longrightarrow \text{P.V.}\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx.$$

Along the contour γ_2, we have $z = Re^{\mathbf{i}t}$ for $t \in [0, \pi]$, thus

$$\int_{\gamma_2} f(z)\,dz = \int_0^\pi \frac{R^2 e^{2\mathbf{i}t}}{R^6 e^{6\mathbf{i}t}+1} \cdot R\mathbf{i}e^{\mathbf{i}t}dt = \int_0^\pi \frac{R^3\mathbf{i}e^{3\mathbf{i}t}\,dt}{R^6 e^{6\mathbf{i}t}+1}.$$

Since $\left|R^6 e^{6\mathbf{i}t} + 1\right| \geq R^6 - 1$ and $\left|R^3\mathbf{i}e^{3\mathbf{i}t}\right| = R^3$, we have

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \int_0^\pi \frac{R^3\,dt}{R^6-1} = \frac{\pi R^3}{R^6-1} \longrightarrow 0.$$

Note that in Equation ($*$) above, LHS does not depend on R while RHS depends on R. This guarantees that for $R \to \infty$, the limit of RHS exists. Hence,

$$\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$$

17.2.2 Integrals from Fourier Analysis

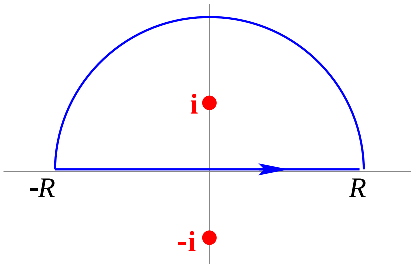
In Fourier Analysis, we often come across integrals of the form

$$\int_{-\infty}^{\infty} f(x)\cos(ax)\,dx, \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\sin(ax)\,dx.$$

These can also be done by the method of residue.

Example 17.4. To evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(3x)\,dx}{(x^2+1)^2}$, we consider the function

$$f(z) = \frac{e^{3\mathbf{i}z}}{(x^2+1)^2} = \frac{e^{3\mathbf{i}z}}{(z+\mathbf{i})^2(z-\mathbf{i})^2}.$$



Take the contour $\Gamma = (\gamma_1, \gamma_2)$, where γ_1 is parametrized by $z(t) = t$ for $t \in [-R, R]$ and γ_2 by $z(t) = Re^{\mathbf{i}t}$ for $t \in [0, \pi]$. Then

$$\int_{\gamma_1} f = \int_{-R}^{R} \frac{\cos(3x)\,dx}{(x^2+1)^2} + \mathbf{i}\int_{-R}^{R} \frac{\sin(3x)\,dx}{(x^2+1)^2}\,,$$

$$\left|\int_{\gamma_2} f\right| \leq \int_0^{\pi} \frac{\left|e^{3\mathbf{i}Re^{\mathbf{i}t}}\right|}{(R^2-1)^2} \cdot \left|R\mathbf{i}e^{\mathbf{i}t}\right|\,dt = \int_0^{\pi} \frac{R}{(R^2-1)^2}\left|e^{3R(\mathbf{i}\cos t - \sin t)}\right|\,dt$$

$$= \int_0^{\pi} \frac{R}{(R^2-1)^2} e^{-3R\sin t}\,dt \leq \frac{\pi R}{(R^2-1)^2} \qquad \text{because } e^{-3R\sin t} \leq 1\,.$$

From the above, we can conclude that $\displaystyle\int_{-\infty}^{\infty} \frac{e^{3\mathbf{i}x}}{(x^2+1)^2}\,dx = \lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 2\pi\mathbf{i}\mathrm{Res}(f, \mathbf{i})$.

It is clear that $\mathbf{i}$ is a pole of order 2 of $f(z) = \dfrac{\varphi(z)}{(z-\mathbf{i})^2}$ where $\varphi(z) = \dfrac{e^{3\mathbf{i}z}}{(z+\mathbf{i})^2}$. Thus,

$$\mathrm{Res}(f, \mathbf{i}) = \lim_{z\to\mathbf{i}} \varphi'(z) = -\mathbf{i}e^{-3}\,.$$

To summarize,

$$\int_{-\infty}^{\infty} \frac{e^{3\mathbf{i}x}}{(x^2+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx + \mathbf{i}\int_{-\infty}^{\infty} \frac{\sin(3x)}{(x^2+1)^2}\,dx = 2\pi\mathbf{i}(-\mathbf{i}e^{-3}) = \frac{2\pi}{e^3}\,.$$

The result comes from comparing the real and imaginary parts in the above equation.

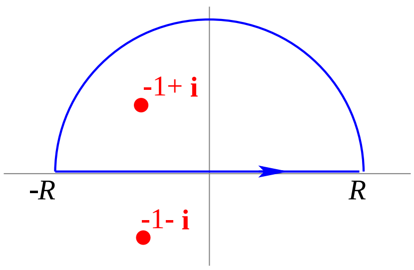
Note that a crucial step is about the estimate

$$\left|\int_{\gamma_2} f\right| \leq \frac{R \text{ due to arc length } \gamma_2}{R^4 \text{ due to the function } f}\,.$$

This is true here only because of the high power of the denominator in f. In the future, we may come across difficult cases.

EXAMPLE 17.5. Evaluate the improper integral P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{x^2+2x+2}$. Naturally, let

$$g(z) = \frac{ze^{\mathbf{i}z}}{z^2+2z+2} = \frac{ze^{\mathbf{i}z}}{(z+1-\mathbf{i})(z+1+\mathbf{i})}\,.$$



Similar to before, we have $\int_{\gamma_1} g(z)\,dz \longrightarrow \text{P.V.}\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{x^2+2x+2}$. Moreover, we have

$$\int_\Gamma g(z)\,dz = 2\pi\mathbf{i}\,\mathrm{Res}(g,-1+\mathbf{i}) = 2\pi\mathbf{i}\,\frac{(-1+\mathbf{i})e^{-\mathbf{i}-1}}{-1+\mathbf{i}+1+\mathbf{i}} = \frac{\pi(-1+\mathbf{i})}{e^{1+\mathbf{i}}}\,.$$

The key to the solution is to estimate $\left|\int_{\gamma_2} g(z)\,dz\right|$ in terms of R. Note that

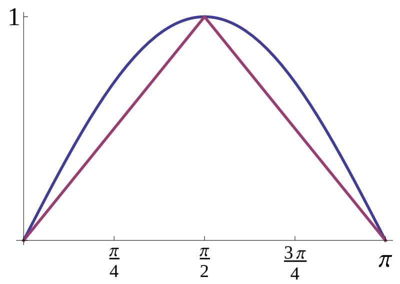
$$\begin{aligned}\left|\int_{\gamma_2} g(z)\,dz\right| &\le \int_0^\pi \frac{\left|Re^{\mathbf{i}t}\right|\cdot\left|e^{\mathbf{i}\left(Re^{\mathbf{i}t}\right)}\right|}{\left|(Re^{\mathbf{i}t}+1-\mathbf{i})(Re^{\mathbf{i}t}+1+\mathbf{i}\right|}\left|R\mathbf{i}e^{\mathbf{i}t}\right|\,dt \\ &= \int_0^\pi \frac{R^2\,e^{-R\sin t}}{|R-|1-\mathbf{i}||\cdot|R-|1+\mathbf{i}||} \le \frac{\pi R^2}{\left|R-\sqrt{2}\right|^2} \qquad \text{which } \not\to 0\,.\end{aligned}$$

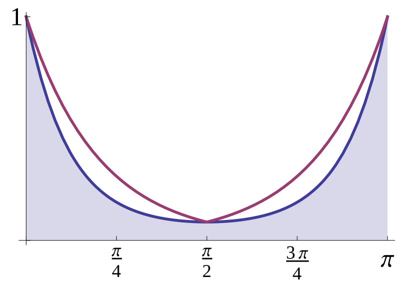
Therefore, simply using $e^{-R\sin t}\le e^0 = 1$ above is not enough to get our desired conclusion. We will get a better estimate of it below.

Let us consider the inequality above

$$\left|\int_{\gamma_2} g(z)\,dz\right| \le \int_0^\pi \frac{R^2\,e^{-R\sin t}}{|R-|1-\mathbf{i}||\cdot|R-|1+\mathbf{i}||} = \frac{R^2}{\left|R-\sqrt{2}\right|^2}\int_0^\pi e^{-R\sin t}\,dt\,.$$

Previously, we used $e^{-R\sin t}\le 1$ to conclude that $\int_0^\pi e^{-R\sin t}\,dt \le \pi$. We need to improve this estimate and the following pictures may be illustrative.





The blue curve in the left hand picture is the graph of $\sin t$ and the one in the right hand picture is $e^{-R\sin t}$. The integral $\int_0^\pi e^{-R\sin t}\,dt$ is the area of the shaded region, which is clearly much less than π (the area of the rectangle). We would try to control the shaded area by R.

Consider the triangle in the right hand picture, which is given by

$$t\mapsto\varphi(t) = \begin{cases}2t/\pi & t\in\left[0,\frac{\pi}{2}\right]\\ -2(t-\pi)/\pi & t\in\left[\frac{\pi}{2},\pi\right]\end{cases}.$$

Clearly, $\varphi(t)\le\sin t$ and thus $e^{-R\sin t}\le e^{-R\varphi(t)}$. Therefore

$$\int_0^\pi e^{-R\sin t}\,dt \le \int_0^\pi e^{-R\varphi(t)}\,dt = 2\int_0^{\pi/2} e^{-2Rt/\pi}\,dt = \frac{\pi}{R}\left(1-e^{-R}\right) \le \frac{\pi}{R}\,.$$

Consequently, $\text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin x\,dx}{x^2+2x+2} = \mathrm{Re}\left(\frac{\pi(-1+\mathbf{i})}{e^{1+\mathbf{i}}}\right) = \frac{\pi}{e}\left[\sin(1)-\cos(1)\right]$.

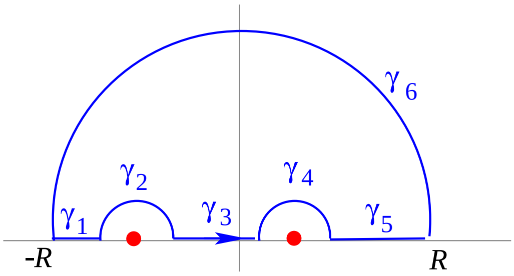
17.2.3 Getting around a Simple Pole

In all the example above, the singularities are in the complex plane but not on the real line. Thus, we can use residue to do the calculation. If there is a singularity on the real line, then the integral may blow up. We will see below that the only workable case is that the singularity is a ***simple pole***.

EXAMPLE 17.6. To find P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2+\pi x-2\pi^2}$. Following the same technique as before, let

$$g(z)=\frac{e^{\mathbf{i}z}}{z^2+\pi z-2\pi^2}=\frac{e^{\mathbf{i}z}}{(z-\pi)(z+2\pi)}\,.$$

Oop! The singularity set $\{-2\pi,\pi\}\subset\mathbb{R}$. So, it is impossible to draw a straight line from $-R$ to R without passing through the singularities. Therefore, we use small indented circles to get around the singularities, $\Gamma=(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6)$ as shown below.



More precisely, the small circles (in negative orientation) can be given by

$$-\gamma_2:\ z=-2\pi+\delta e^{\mathbf{i}t},\quad t\in[0,\pi];\qquad \text{and}\qquad -\gamma_4:\ z=\pi+\varepsilon e^{\mathbf{i}t},\quad t\in[0,\pi]\,.$$

It is expected that $\displaystyle\int_{\gamma_1} g(z)\,dz+\int_{\gamma_3} g(z)\,dz+\int_{\gamma_5} g(z)\,dz$ will give what we want when $R\to\infty$, $\delta\to 0$, and $\varepsilon\to 0$. Moreover, by the same argument as in the previous examples, we have

$$\left|\int_{\gamma_6} g(z)\,dz\right|\le\frac{\pi R}{(R-\pi)(R-2\pi)}\longrightarrow 0,\qquad \text{and}\qquad \int_\Gamma g(z)\,dz=0\,.$$

The crucial step is to find the contour integral along γ_2 and γ_4. That is

$$\int_{\gamma_2} g(z)\,dz=-\int_0^\pi \frac{e^{\mathbf{i}\left(-2\pi+\delta e^{\mathbf{i}t}\right)}}{(-2\pi+\delta e^{\mathbf{i}t}-\pi)\,\delta e^{\mathbf{i}t}}\,\delta\mathbf{i}e^{\mathbf{i}t}\,dt\longrightarrow-\int_0^\pi\frac{e^{-2\pi\mathbf{i}}}{-3\pi}\,\mathbf{i}\,dt=\frac{\mathbf{i}}{3\pi}\,,$$
$$\int_{\gamma_4} g(z)\,dz=-\int_0^\pi \frac{e^{\mathbf{i}\left(\pi+\varepsilon e^{\mathbf{i}t}\right)}}{\varepsilon e^{\mathbf{i}t}\,(\pi+\varepsilon e^{\mathbf{i}t}+2\pi)}\,\varepsilon\mathbf{i}e^{\mathbf{i}t}\,dt\longrightarrow-\int_0^\pi\frac{e^{\pi\mathbf{i}}}{3\pi}\,\mathbf{i}\,dt=\frac{\mathbf{i}}{3\pi}\,.$$

Hence we have P.V. $\displaystyle\int_{-\infty}^{\infty}\frac{\cos x\,dx}{x^2+\pi x-2\pi^2}=0$ and P.V. $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x^2+\pi x-2\pi^2}=\frac{-2}{3\pi}$.
Note that in the above, the cancellation of δ and ε works in both cases because both -2π and π are simple poles of the function. This is indeed related to the convergence of integrating an unbounded function discussed in mathematical analysis.