THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 16 Residues

16.1 Singularities and Zeros

For now, we have known the following facts.

- If f is analytic on *and inside* a closed contour Γ, then $\displaystyle\int_\Gamma f = 0$.
- If f is analytic on Γ and except a few points, $z_k, k = 1, \ldots, n$ inside it, then

$$\int_\Gamma f = \sum_{k=1}^{n} \int_{C_k} f \qquad \text{where } C_k \text{ is a small circle with center } z_k.$$

- If f is analytic on $B(z_0, \varepsilon) \setminus \{z_0\}$, then f has a Laurent Series on $B(z_0, \varepsilon)$,

$$f(z) = \sum_{k=2}^{\infty} \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-1}}{z-z_0} + a_0 + \sum_{\ell=1}^{\infty} a_\ell (z-z_0)^\ell .$$

If C_0 is a small circle with center z_0, by direct calculation, we have

$$\int_{C_0} f = \cdots + 0 + 0 + 2\pi\mathbf{i}\, a_{-1} + 0 + 0 + \cdots .$$

The term a_{-1} is the most important in the Laurent Series and it often can be found by other methods. Thus, we specially focus on it.

16.1.1 Isolated Singularities

Observe that the above discussion only works if there are finitely many singularities *inside* Γ.

Example 16.1. Let $f(z) = 1/\sin\left(\frac{\pi}{z}\right)$. The set of singularities is $\{0\} \cup \{\, 1/n : n \in \mathbb{Z}\,\}$. Clearly, if a contour Γ contains 0 in its inside, there will be infinitely many singularities inside. Moreover, at $z_0 = 0$, the condition for Laurent Series does not hold.

Definition 16.2. A point $z_0 \in \mathbb{C}$ is an *isolated singularity* for a function f if f is analytic on $B(z_0, \varepsilon) \setminus \{z_0\}$ for some $\varepsilon > 0$.

Then, at an isolated singularity z_0, there is a Laurent Series for the function f. The singularity is classified into three types as follows.

- It is *removable* if the Laurent Series does not contain any term of negative powers, i.e., it is indeed a Power Series. A typical example is $\dfrac{\sin z}{z}$, $z_0 = 0$.
- It is *essential* if the Laurent Series has infinitely many terms of negative powers. A typical example is $e^{1/z}$, $z_0 = 0$.

- It is a *pole* of orde p if the smallest negative power is $-p$ in the Laurent Series, i.e.,

$$f(z) = \sum_{k=1}^{p} \frac{a_{-k}}{(z-z_0)^k} + \sum_{\ell=0}^{\infty} a_\ell (z-z_0)^\ell, \qquad a_{-p} \neq 0\,.$$

If $p = 1$, it is called a *simple pole*.

EXAMPLE 16.3. Let $f(z) = \dfrac{(e^z - 1)\sin z}{z^3(z-1)^2(z+1)(z-\pi)}$. The singularity set is $\{\,-1, 0, 1, \pi\mathbf{i}\,\}$. Most of the times, we can simply look at the function and determine the order of the pole. For $z_0 = -1$, we see that

$$g(z) = \frac{(e^z - 1)\sin z}{z^3(z-1)^2(z-\pi)}$$

is analytic in a small ball at center -1 and $g(-1) \neq 0$. Therefore, $g(z)$ is a power series of $(z+1)$ near -1 and the constant term is $g(-1)$. Then the Laurent Series of f becomes

$$\frac{g(-1)}{z+1} + \text{power series of } (z+1).$$

For this reason, we know that -1 is a simple pole of f. Similarly, the point $z_0 = 1$ is a pole of order 2. When we consider the singularity π, it is not a simple pole because $\sin(\pi) = 0$. In fact,

$$\sin z = -\sin(z-\pi) = -(z-\pi)\left[1 - \frac{1}{3!}(z-\pi)^2 + \frac{1}{5!}(z-\pi)^4 - \cdots\right].$$

Thus, π is indeed a removable singularity. For $z_0 = 0$, observe the power series of $e^z - 1$ and $\sin z$, one knows that $z_0 = 0$ is a simple pole also.

16.1.2 Residue

As we have mentioned, if C_0 is a small circle with center at an isolated singularity z_0, then the Laurent Series can be integrated term by term and the coefficient a_{-1} in the series most crucial because

$$\int_{C_0} f(z)\,dz = 2\pi\mathbf{i}a_{-1}\,.$$

For this, we define the *Residue of f at z_0* to be a_{-1} of the Laurent Series at z_0. It is denoted by

$$\mathrm{Res}_{z_0} f, \qquad \text{or} \quad \mathrm{Res} f(z_0), \qquad \text{or} \quad \mathrm{Res}(f, z_0)\,.$$

EXAMPLE 16.4. Let $f(z) = \dfrac{2z+1}{z^3(z^2+1)}$. We have poles at 0, $\pm\mathbf{i}$. By partial fraction,

$$f(z) = \frac{1}{z^3} + \frac{2}{z^2} - \frac{1}{z} + \frac{\frac{1}{2} - \mathbf{i}}{z + \mathbf{i}} + \frac{\frac{1}{2} + \mathbf{i}}{z - \mathbf{i}}\,.$$

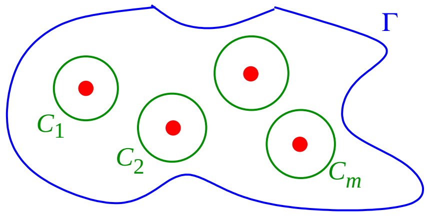
This shows that (give the argument yourself)

$$\mathrm{Res}(f, 0) = -1; \qquad \mathrm{Res}(f, -\mathbf{i}) = \frac{1}{2} - \mathbf{i}; \qquad \mathrm{Res}(f, \mathbf{i}) = \frac{1}{2} + \mathbf{i}\,.$$

THEOREM 16.5 (Cauchy Residue Theorem). *Let Γ be a simple closed (positively oriented) contour with bounded complement component S. If f is analytic on Γ and $S \setminus \{\, z_1, \ldots, z_m \,\}$ then*

$$\int_\Gamma f = 2\pi\mathbf{i} \sum_{k=1}^{m} Res(f, z_k)\,.$$

It follows very easily from Cauchy-Goursat Theorem. The function f is analytic on the complement $S \setminus (\Gamma, -C_1, \ldots, -C_m)$ and the integral is zero.



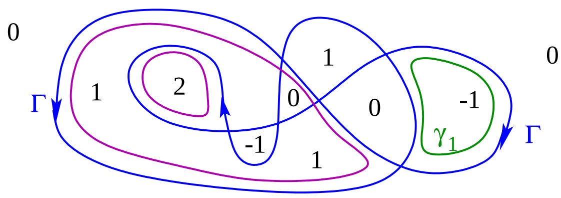
Thus, we have $\displaystyle\int_\Gamma f(z)\,dz = \sum_{k=1}^m \int_{C_k} f(z)\,dz = 2\pi\mathbf{i}\sum_{k=1}^m \mathrm{Res}(f, z_k)$. In fact, there is a more general version that does not require the contour to be simple.

THEOREM 16.6 (General Residue Theorem). *Let $f:\ \Omega \subset \mathbb{C} \to \mathbb{C}$ be analytic except at the points $z_1, \ldots, z_m \in \Omega$. If Γ is a closed contour in Ω, then*

$$\int_\Gamma f = 2\pi\mathbf{i}\sum_{k=1}^m \nu(\Gamma, z_k)\, Res(f, z_k)\,,$$

where $\nu(\Gamma, z_k)$ is the winding number of Γ about z_k.

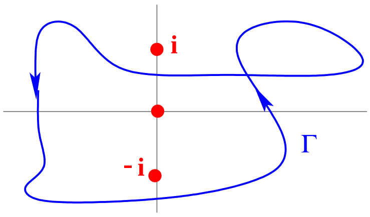
Intuitively, the winding number counts how many times the contour Γ circles around a point. So, it two points are in the same complement component of $\Omega \setminus \Gamma$, the winding number is the same. Let us use the following picture to illustrate the winding number.



The proof is simply by decomposing Γ into several simple closed contours. For example, on the right hand side of the picture, if a singularity lies in the component marked -1, only the green contour γ_1 has will give a residue and the curve is negatively oriented. In the middle of the picture, if z_k lies in the component marked 2, the two violet contours (positively oriented) will contribute to the residue.

EXAMPLE 16.4. This is an example given above, that $f(z) = \dfrac{2z+1}{z^3(z^2+1)}$ and

$$\mathrm{Res}(f, 0) = -1; \qquad \mathrm{Res}(f, -\mathbf{i}) = \frac{1}{2} - \mathbf{i}; \qquad \mathrm{Res}(f, \mathbf{i}) = \frac{1}{2} + \mathbf{i}\,.$$

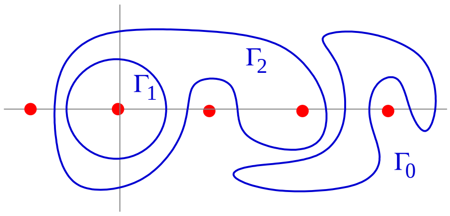


For the contour on the left,

$$\nu(\Gamma, 0) = \nu(\Gamma, -\mathbf{i}) = 1\,, \quad \text{and} \quad \nu(\Gamma, \mathbf{i}) = 0\,.$$

Thus, $\displaystyle\int_\Gamma f(z)\,dz = 2\pi\mathbf{i}\left(-1 + \frac{1}{2} - \mathbf{i}\right)$.

Example 16.7. Let $f(z) = \dfrac{1}{e^{\mathbf{i}z} - 1}$, which has singularities at $\{\, 2k\pi : \ k \in \mathbb{Z} \,\}$. Find it integral along the following contours.



First, it is easy that $\displaystyle\int_{\Gamma_0} f = 0$ (give your own reason).

Second, at $z_0 = 0$, we may write the power series $e^{\mathbf{i}z} - 1 = z\left(\mathbf{i} - \dfrac{z}{2!} - \dfrac{\mathbf{i}z^2}{3!} + \cdots\right)$ and

$$f(z) = \frac{1}{z} \cdot \frac{1}{\mathbf{i} - \frac{z}{2!} - \frac{\mathbf{i}z^2}{3!} + \cdots} = \frac{1}{z}\left[-\mathbf{i} - \frac{z}{2!} + \left(\frac{1}{3!} - \frac{1}{(2!)^2}\right) z^2 + \cdots\right] = \frac{g(z)}{z}\,.$$

From this, we can see that $\operatorname{Res}(f, 0) = -\mathbf{i}$ and so $\displaystyle\int_{\Gamma_1} f = 2\pi$. The same result also follows from

$$\int_{\Gamma_1} f(z)\, dz = \int_{\Gamma_1} \frac{g(z)}{z}\, dz = 2\pi\mathbf{i}\, g(0) = 2\pi, \qquad \text{by Cauchy Integral Formula.}$$

We will leave that $\displaystyle\int_{\Gamma_2} f = 4\pi$ as an exercise.

16.2 Poles and Zeros

Recall that a function f has a pole of order p at z_0 if it is analytic on $B(z_0, \varepsilon) \setminus \{z_0\}$ and its Laurent Series is of the form

$$f(z) = \sum_{k=1}^{p} \frac{a_{-k}}{(z - z_0)^k} + \sum_{\ell=0}^{\infty} a_\ell (z - z_0)^\ell\,, \qquad \text{where } a_{-p} \neq 0.$$

An obvious example that will create a pole of order p is

$$f(z) = \frac{\text{analytic}}{(z - z_0)^p} = \frac{\sum_{k=0}^{\infty} a_k (z - z_0)^k}{(z - z_0)^p} = \sum_{k=1}^{p} \frac{a_{p-k}}{(z - z_0)^k} + \sum_{\ell=0}^{\infty} a_{p+1+\ell} (z - z_0)^\ell\,.$$

Question. What is the general pattern of such a pole? Is the above the only situation?

In order to study the situation, we start with the study of zeros. First of all, if h is analytic on $B(z_0, \varepsilon)$, it has a power Series. Therefore, $h(z_0) = 0$ really means $a_0 = 0$ and possible more zero terms.

Definition 16.8. Let h be analytic on $B(z_0, \eta)$. The point z_0 is called a *zero of order m* of h if

$$h(z_0) = 0 = h'(z_0) = \cdots = h^{(m-1)}(z_0), \qquad \text{and} \quad h^{(m)}(z_0) \neq 0\,.$$

Equivalently, h has a Taylor Series at z_0 of the form

$$h(z) = (z - z_0)^m \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m} = (z - z_0)^m \varphi(z)\,.$$

The function $\varphi(z)$ above is analytic on $B(z_0, \eta)$ and $\varphi(z_0) \neq 0$. Then by continuity of φ and taking a smaller radius $0 < \varepsilon < \eta$, we may assume that $\varphi \neq 0$ on $B(z_0, \varepsilon)$. Since $(z - z_0) \neq 0$ on $B(z_0, \varepsilon) \setminus \{z_0\}$, we have $h(z) \neq 0$ on the punctured ball. Hence, z_0 is an isolated zero of h. From the above argument, we see that non-constant analytic functions only have isolated zeros.

THEOREM 16.9. *Let f be analytic on Ω and $z_0 \in \Omega$. If $f(z_0) = 0$ and there exists a sequence $z_n \to z_0$ such that $f(z_n) = 0$, then $f \equiv 0$ on Ω. In particular, if $f\,|_L \equiv 0$ where $L \subset \Omega$ is a line or a continuous arc, then $f \equiv 0$ on Ω.*

Now, we can describe the situation of a pole in the above language. The proof is trivial.

THEOREM 16.10. *If $h(z), q(z)$ are analytic on a ball $B(z_0, \eta)$, where z_0 is a zero of order m for h and one of order $m + p$ for q, then z_0 is a pole of order p for $f(z) = h(z)/q(z)$. The converse is also true, i.e., any pole of order p can be written into this form.*

16.2.1 Calculating Residue at Poles

Since calculating residue is so helpful in finding contour integral, this becomes a central matter. For an essential singularity, there is no short cut. The only way is to write the Laurent Series and get the a_{-1} term.

On the other hand, there are good methods for the residue at a pole. The methods are usually neat and tidy for a simple pole. And the understanding of such methods will be helpful to multiple pole.

Let z_0 be a simple pole of the function f. Then we have

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{\ell=0}^{\infty} a_\ell (z - z_0)^\ell\,.$$

Then, multiplying $(z - z_0)$ to both sides, we have

$$f(z)(z - z_0) = a_{-1} + (z - z_0) \sum_{\ell=0}^{\infty} a_\ell (z - z_0)^\ell\,.$$

Thus, $\lim\limits_{z \to z_0} f(z)(z - z_0) = a_{-1} = \mathrm{Res}(f, z_0)$. Note that although we write $\lim\limits_{z \to z_0}$, the actual calculation is more like substituting z_0 to both sides.

EXAMPLE 16.11. Consider $f(z) = \dfrac{(e^{\pi z} - 1)\sin(2\pi z)}{z^3(z^2 + 1)}$ which has poles $0, \mathbf{i}, -\mathbf{i}$; all of order 1. Then

$$\mathrm{Res}(f, \mathbf{i}) = \lim_{z \to \mathbf{i}} \frac{(e^{\pi z} - 1)\sin(2\pi z)}{z^3(z + \mathbf{i})} = \frac{(e^{\pi \mathbf{i}} - 1)\sin(2\pi \mathbf{i})}{\mathbf{i}^3(\mathbf{i} + \mathbf{i})} = -\sin(2\pi \mathbf{i})\,.$$

In the process of finding $\mathrm{Res}(f, 0)$, one needs to find limit instead of substitution.

The above method can easily be extended to the case of multiple pole at z_0, in which case

$$f(z) = \sum_{k=2}^{p} \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-1}}{z-z_0} + \sum_{\ell=0}^{\infty} a_\ell (z-z_0)^\ell \,.$$

After multiplying both sides with $(z-z_0)^p$, we have

$$f(z)(z-z_0)^p = a_{-p} + a_{-p+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{p-1} + (z-z_0)^p \sum_{\ell=0}^{\infty} a_\ell (z-z_0)^\ell \,.$$

Now, simply taking $z \to z_0$ will not give a_{-1}. So, we need to differentiate the equation $(p-1)^{\text{th}}$ times first. As a result, $\mathrm{Res}(f, z_0)$ comes from this,

$$\lim_{z\to z_0} \frac{d^{p-1}}{dz^{p-1}} \left[f(z)(z-z_0)^p \right] = (p-1)!\, a_{-1} + 0 + 0 + \cdots \,.$$

In the case that z_0 is a simple pole, as mentioned in how a pole can occur, we know that

$$f(z) = \frac{h(z)}{q(z)} = \frac{h(z)}{(z-z_0)\psi(z)}, \qquad h(z_0) \neq 0, \psi(z_0) \neq 0 \,.$$

Since both h and ψ are analytic, we may write them into power series,

$$f(z) = \frac{1}{z-z_0} \cdot \frac{\sum_{k=0}^{\infty} a_k (z-z_0)^k}{\sum_{\ell=0}^{\infty} b_k (z-z_0)^\ell} = \frac{1}{z-z_0} \sum_{j=0}^{\infty} c_j (z-z_0)^j \,.$$

It follows that

$$\mathrm{Res}(f, z_0) = c_0 = \frac{a_0}{b_0} = \frac{h(z_0)}{\psi(z_0)} = \frac{h(z_0)}{q'(z_0)} \,.$$

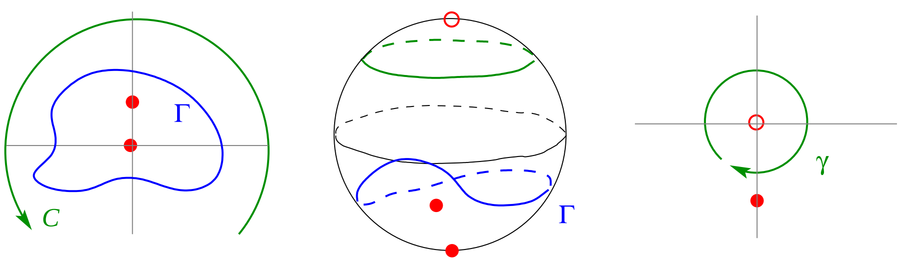
For example, $\mathrm{Res}\left[\dfrac{e^z}{\cos z}, \dfrac{\pi}{2}\right] = \dfrac{e^{\pi/2}}{-\sin(\pi/2)} = -e^{\pi/2}$.

16.2.2 All Singularities and Infinity

Let us start with an example that $f(z) = \dfrac{3+\mathbf{i}}{z(z+\mathbf{i})} = \dfrac{1}{z} + \dfrac{2}{z+\mathbf{i}}$ by partial fraction. Therefore, if Γ is a contour containing ***all*** the singularities inside,

$$\int_\Gamma f = 2\pi\mathbf{i}(1+2) = 6\pi\mathbf{i} \,.$$

Now, as Γ contains all the singularities, it can be replaced by a circle C of larger and larger radius (left picture below). When drawn on the sphere by stereographic projection, this large circle becomes a small circle surrounding the north pole (center picture below).



Note that the mapping $w = 1/z$ invert the positions of 0 and ∞ (south and north pole). Then C becomes a *negative oriented* small circle γ around the origin (right picture above). Mathematically,

$$f(z) = \frac{3z + \mathbf{i}}{z(z + \mathbf{i})} = \frac{3/w + \mathbf{i}}{\frac{1}{w}\left(\frac{1}{w} + \mathbf{i}\right)} = \frac{w(3 + \mathbf{i}w)}{1 + \mathbf{i}w}\,.$$

Apparently, in w-plane, f has only a singularity $-\mathbf{i}$, which is clearly the image of $\mathbf{i}$ under $w = 1/z$. And the mapping sends 0 to ∞. However, when we do the integral, there will be one more singularity 0, which is the original ∞ under $w = 1/z$.

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \int_\Gamma \frac{w(3 + \mathbf{i}w)}{1 + \mathbf{i}w}\,d\left(\frac{1}{w}\right) = \int_\gamma \frac{w(3 + \mathbf{i}w)}{1 + \mathbf{i}w} \cdot \frac{-1}{w^2}\,dw \\
&= \int_\gamma \frac{-(3 + \mathbf{i}w)}{w(1 + \mathbf{i}w)}\,dw = \int_\gamma -\left(\frac{3}{w} + \frac{-2}{w - \mathbf{i}}\right)\,dw \\
&= \int_{-\gamma} \left(\frac{3}{w} + \frac{-2}{w - \mathbf{i}}\right)\,dw = 2\pi\mathbf{i}(3)\,, \qquad \text{since } \mathbf{i} \text{ is outside } \gamma.
\end{aligned}$$

The above calculation shows that

$$\mathrm{Res}\,[f(z), 0] + \mathrm{Res}\,[f(z), \mathbf{i}] = -\mathrm{Res}\left[\frac{-1}{w^2} f\left(\frac{1}{w}\right), 0\right]\,.$$

This phenomenon is indeed true generally and it is expressed below. It provides a method for us to find one residue instead of residues at many singularities.

Theorem 16.12. *Let f be analytic on $\mathbb{C} \setminus \{\, z_1, z_2, \ldots, z_m \,\}$ and Γ be a contour containing all singularities inside. Then*

$$\sum_{k=1}^{m} \mathit{Res}\,[f(z), z_k] = -\mathit{Res}\left[\frac{-1}{w^2} f\left(\frac{1}{w}\right), 0\right]\,.$$

For this reason, we define $\mathrm{Res}(f, \infty) = \mathrm{Res}\left[\frac{-1}{w^2} f\left(\frac{1}{w}\right), 0\right]$. Then we can simply write

$$\int_\Gamma f(z)\,dz = -2\pi\mathbf{i}\,\mathrm{Res}(f, \infty)\,.$$

Proof. Let $R > 0$ be a large radius such that $\{\, z_1, z_2, \ldots, z_m \,\} \subset B(0, R)$. Then f is analytic on the annulus $A(0, R, \infty)$ and so it has a Laurent Series with center 0, i.e.,

$$\sum_{k=1}^{\infty} \frac{a_{-k}}{z^k} + \sum_{\ell=0}^{\infty} a_\ell z^\ell, \qquad \text{where } a_{-k} = \int_C \frac{f(\zeta)}{(\zeta - 0)^{-k+1}}\,d\zeta\,.$$

In particular, $a_{-1} = \frac{1}{2\pi\mathbf{i}} \int_C f(\zeta)\,d\zeta = \frac{1}{2\pi\mathbf{i}} \int_\Gamma f(z)\,dz = \sum_{k=1}^{m} \mathrm{Res}(f, z_k)$.

On the other hand,

$$\begin{aligned}
f\left(\frac{1}{w}\right) \cdot \frac{-1}{w^2} &= \frac{-1}{w^2} \cdot \left(\sum_{k=1}^{\infty} \frac{a_{-k}}{(1/w)^k} + \sum_{\ell=0}^{\infty} a_\ell \left(\frac{1}{w}\right)^\ell\right) = \frac{-1}{w^2} \cdot \left(\sum_{k=1}^{\infty} a_{-k} w^k + \sum_{\ell=0}^{\infty} \frac{a_\ell}{w^\ell}\right) \\
&= \sum_{\ell=1}^{\infty} \frac{-a_\ell}{w^{\ell+2}} + \frac{-a_0}{w^2} + \frac{-a_{-1}}{w} - \sum_{k=2}^{\infty} a_{-k} w^{k-2}\,.
\end{aligned}$$

From this we see that $-a_{-1} = \mathrm{Res}\left[\frac{-1}{w^2} f\left(\frac{1}{w}\right), 0\right] = \mathrm{Res}(f, \infty)$. ∎