THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 13 Properties of Analytic Functions

13.1 Center "Less" than Average Around

By the Mean Value Property of an analytic function, the value at a point is related to the values of its surrounding. There are stronger impact when we consider the derivatives.

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be analytic. Let $z_0 \in \Omega$ and take a radius $R > 0$ such that the closed ball $D = \{ z \in \mathbb{C} : \ |z - z_0| \le R \} \subset \Omega$. Also, let $C = \partial D$ be the boundary circle, i.e., with center z_0 and radius R. By continuity of f on C, there exists $M_R \in \mathbb{R}$ such that $|f(\zeta)| \le M_R$ for all $\zeta \in C$. That is, $M_R = \max\limits_{\zeta \in C} |f(\zeta)|$.

Parametrize the circle C by $Re^{\mathbf{i}t}$ for $t \in [0, 2\pi]$. Then by the Cauchy Integral Formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\,,$$
$$\left| f^{(n)}(z_0) \right| \le \frac{n!}{2\pi} \int_C \frac{M_R}{|\zeta - z_0|^{n+1}}\, |d\zeta| = \frac{n!}{2\pi} \int_C \frac{M_R}{R^{n+1}}\, |d\zeta| = \frac{n!\, M_R}{R^n}\,.$$

Question. What if R can be very very big?

Note that we require $D = \{ z \in \mathbb{C} : \ |z - z_0| \le R \} \subset \Omega$ above. If R can be very very big, the only possibility is that $\Omega = \mathbb{C}$, i.e., f is ***entire.*** Then the inequality is true for all z_0 and R,

$$\left| f^{(n)}(z_0) \right| \le \frac{n!\, M_R}{R^n}\,. \tag{$\dagger$}$$

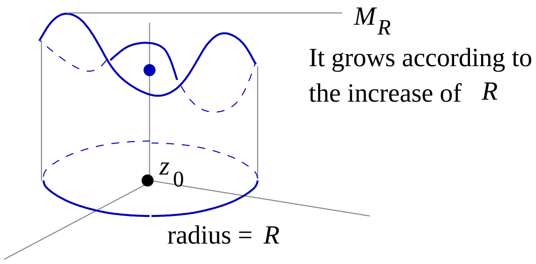
The quantities R^n in the denominator and possibly M_R are growing to infinity as R increases. What happens if M_R is not increasing fast enough, or even bounded?

Theorem 13.1 (Liouville). *Any **bounded** entire function $f : \mathbb{C} \to \mathbb{C}$ is a constant function.*

If f is bounded, we have $M_R \le M$. Then $f^{(n)}(z_0) = 0$ for all $1 \le n \in \mathbb{Z}$ and $z_0 \in \mathbb{C}$. Thus, f must be a constant function.

Remark. Usually, f is not bounded and $M_R \to \infty$. Its growth will tell us more. For example, if $M_R \sim R^p$, then $f^{(p+1)} \equiv 0$ and f can only be a degree p polynomial. On the contrary, any non-polynomial entire function must have exponential growth on M_R.

If f is analytic on a closed ball D as above, the value of f is bounded by its surrounding. And if f is non-constant, the maximum of its surrounding is growing as illustrated by the picture.



Question. What if $|f(z_0)|$ is already a maximum?

Since $f(z_0)$ is sort of an average of $f(\zeta)$ over a circle, the above condition is *almost* saying that the average is the maximum. So, we would expect that M_R is a constant independent of R. But then, will $\zeta \mapsto |f(\zeta)|$ be also a constant, or even f itself is a constant? A more mathematical discussion of this phenomenon will be discussed in the next section.

13.2 Controlled by the Boundary

Previously, from the Cauchy Integral Formula, we obtained that the central value of an analytic function is closely related to the values surrounding the central point. Moreover, this is also true for the derivatives of the function. Two important results are derived. One is the Gauss Mean Value Theorem. The other is the inequality

$$\left|f^{(n)}(z_0)\right| \le \frac{n!\,M_R}{R^n}\,, \qquad M_R = \max_{\zeta\in\partial D} |f(\zeta)|\,,$$

whenever f is analytic on the closed ball $D = \{\,\zeta \in \mathbb{C} : \ |\zeta - z_0| \le R\,\}$. This inequality provides very useful information about analytic functions.

Think about the right hand side of the inequality, M_R depends only on f and the radius R, but not on n. If the function f is analytic on $\mathbb{C}$, then R can be arbitrarily large. The denominator $R^n \to \infty$ and so the behavior of M_R is crucial. If M_R is bounded, then we have the Liouville Theorem. If the growth of M_R is slower than R^d, then f is at most a degree d polynomial. Otherwise, M_R should be growing exponentially.

Then, a natural intuition suggests that if M_R is not growing, then the function f is difficult to behave as an analytic function, unless f is in fact a constant.

13.2.1 About the Modulus

The results in this section seem to be about the modulus $|f(z)|$, which can be seen as a continuous function from $\mathbb{R}^2 = \mathbb{C} \to [0,\infty)$. At the end, we may know more about $f(z)$ itself from its modulus $|f(z)|$.

Let f be analytic on a closed ball $D = \overline{B(z_0,R)} = \{\,\zeta \in \mathbb{C} : \ |\zeta - z_0| \le R\,\}$ and denote the boundary of the ball by ∂D, then from the Cauchy Integral Formula with $n = 0$,

$$|f(z_0)| \le M_R = \max_{\zeta\in\partial D} |f(\zeta)|\,.$$

PROPOSITION 13.2. *Moreover, if $|f(z_0)| = M_R$, then $|f|$ is constant on ∂D (later, we will see that f itself is a constant on the whole D).*

Proof. By Mean Value Property, $f(z_0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(z_0 + Re^{\mathrm{i}t})\,dt$. Thus

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + Re^{\mathrm{i}t})\right|\,dt\,.$$

However, on the other hand, $\left|f(z_0 + Re^{\mathbf{i}t})\right| \le M_R = |f(z_0)|$. Therefore

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + Re^{\mathbf{i}t})\right| \, dt \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| \, dt = |f(z_0)| \, .$$

Equivalently, $0 \le \displaystyle\int_0^{2\pi} \left(\left|f(z_0 + Re^{\mathbf{i}t})\right| - |f(z_0)| \right) dt \le 0$. It follows that for all $t \in [0, 2\pi]$, $\left|f(z_0 + Re^{\mathbf{i}t})\right| = |f(z_0)|$, i.e., $|f|$ is a constant function on D $\square$

The same proof can be used to establish an essentially equivalent result.

Corollary 13.3. *If f is analytic on $B(z_0, \varepsilon)$ and $|f(z_0)| = \sup\{|f(\zeta)| : \zeta \in B(z_0, \varepsilon)\}$, then f is a constant on $B(z_0, \varepsilon)$.*

Proof. For any $0 < R < \varepsilon$, the function f is analytic on the closed ball $D = \overline{B(z_0, R)}$. Therefore, from Proposition 14.1, $|f|$ is constantly $|f(z_0)|$ on ∂D. Since R is arbitrary, it follows that $|f(\zeta)| = |f(z_0)|$ for all $\zeta \in B(z_0, \varepsilon)$. Then, by a standard exercise using Cauchy-Riemann Equations, we have f is constantly $f(z_0)$ on $B(z_0, \varepsilon)$. $\square$

This result can be interpreted in the way that if an analytic function attains its maximum modulus $|f|$ at the central point of a ball, then it is a constant function on the whole ball. So, the following picture about $|f|$ is not allowed.