THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 12 Cauchy Integral Formulas

12.1 A Zero at Denominator

Let us first recall three important results, which we will often use in this lesson.

Theorem 11.2 (Cauchy-Goursat). *Let Γ be a simple closed contour with bounded complement compoent S_b such that $\Gamma \cup S_b \subset \Omega$ and $f:\ \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic. Then* $\displaystyle\int_\Gamma f(z)\,dz = 0$.

From the Cauchy-Goursat Theorem, we are able to further derive two useful theorems.

Theorem 11.4. *Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$ are positive oriented simple closed contours such that $\Gamma_1, \ldots, \Gamma_p$ all lie in the bounded complement component of Γ_0 and $B \subset \Omega$ is the region such that $\partial B = \Gamma_0 \cup (-\Gamma_1) \cup \cdots \cup (-\Gamma_p)$. If f is analytic on the domain Ω, then*

$$\int_{\Gamma_0} f(z)\,dz = \sum_{k=1}^{p} \int_{\Gamma_k} f(z)\,dz\,.$$

Theorem 11.5 (Invariance of Deformation). *If Γ_1 and Γ_2 can be deformed smoothly to each other through a region $B \subset \Omega$ where f is analytic on Ω, then* $\displaystyle\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$.

12.1.1 The Zero is Simple

It has been seen that for the integral $\displaystyle\int_\Gamma g(z)\,dz$ where g is a rational function, one may use partial fraction to break down g and consider integrands of the form $\dfrac{A}{z-z_0}$. However, the integrand may not be a rational function. Examples below are slightly more complicated than rational functions,

$$\frac{\sin z}{z(z-1)} = \frac{-\sin z}{z} + \frac{\sin z}{z-1}\,, \qquad \frac{e^z}{z^2-1} = \frac{-e^z/2}{z+1} + \frac{e^z/2}{z-1}\,.$$

Thus, the aim of this section is to deal with the integrals of the form $\displaystyle\int_\Gamma \frac{f(z)}{z-z_0}\,dz$.

Theorem 12.1 (Cauchy Integral Formula). *Let $f:\ \Omega \subset \mathbb{C} \to \mathbb{C}$ be analytic and Γ be a simple closed contour with bounded complement component $S_b \subset \Omega$. Then, for $z_0 \in \mathbb{C} \setminus \Gamma$,*

$$\int_\Gamma \frac{f(z)}{z-z_0}\,dz = \begin{cases} 0 & z_0 \notin S_b \cup \Gamma\,, \\ 2\pi \mathbf{i} f(z_0) & z_0 \in S_b\,. \end{cases}$$

If $z_0 \notin S_b$ then clearly $\dfrac{f(z)}{z-z_0}$ is analytic on Ω and we can simply apply the Cauchy-Goursat Theorem to get $\displaystyle\int_\Gamma \frac{f(z)}{z-z_0}\,dz = 0$.

Let $z_0 \in S_b$ and C_δ be a circle with center z_0 and a small radius $\delta > 0$. Then, the integrand is analytic on the region *between* C_δ and Γ. By either Invariance of Deformation or Theorem 11.4, one has

$$\int_\Gamma \frac{f(z)}{z-z_0}\,dz = \int_{C_\delta} \frac{f(z)}{z-z_0}\,dz = \int_{C_\delta} \frac{f(z_0)}{z-z_0}\,dz + \int_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$
$$= 2\pi\mathbf{i} f(z_0) + \int_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz\,.$$

On the other hand, f is differentiable at z_0, thus for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon\,.$$

By taking a sufficiently small radius $\delta > 0$, the last integral above is controlled by

$$\left|\int_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \int_{C_\delta} \left(\left|f'(z_0)\right| + \varepsilon\right)\,|dz| \le 2\pi\varepsilon\,\left(\left|f'(z_0)\right| + \varepsilon\right)\,.$$

Since ε is arbitrary, we have

$$\int_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0, \qquad \text{and so} \quad \int_\Gamma \frac{f(z)}{z-z_0}\,dz = 2\pi\mathbf{i} f(z_0)\,.$$

12.1.2 Intuition from Examples

Before we go on to further theories, let us look at more examples and get some feeling.

EXAMPLE 12.2. Consider $\displaystyle\int_\Gamma \frac{f(z)}{z-1}\,dz$ where Γ is simple closed positively oriented with $z_0 = 1$ in its bounded complement and $f(z) = z^3 - z^2 - 2z + 5$.

We may use the Cauchy Integral Formula and get the answer $2\pi\mathbf{i}f(1) = 6\pi\mathbf{i}$. Or, we may work out the partial fraction and take a small circle C at center $z_0 = 1$,

$$\frac{f(z)}{z-1} = z^2 - 2 + \frac{3}{z-1} \qquad \text{thus} \quad \int_\Gamma \frac{f(z)}{z-1}\,dz = 0 + \int_C \frac{3}{z-1}\,dz = 6\pi\mathbf{i}\,. \tag{12.2}$$

EXAMPLE 12.3. Let us change the problem to $\displaystyle\int_\Gamma \frac{f(z)}{(z-1)^2}\,dz$ where Γ and f are the same.

Now, the Cauchy Integral Formula does not work in this case! Of course, we may still work by the partial fraction

$$\frac{f(z)}{(z-1)^2} = z + 1 + \frac{-z+4}{(z-1)^2} = z + 1 + \frac{-1}{z-1} + \frac{3}{(z-1)^2}\,. \tag{12.3}$$

Then, we may take a small circle C with center $z_0 = 1$ to have

$$\int_\Gamma \frac{f(z)}{(z-1)^2}\,dz = 0 + \int_C \frac{-1}{z-1}\,dz + \int_C \frac{3}{(z-1)^2}\,dz = 0 - 2\pi\mathbf{i} + 0\,.$$

Unfortunately, this method of partial fraction does not work for ***non-polynomial*** analytic function $f(z)$. But, we may see some hint by re-writing Equations (12.2) and (12.3).

For $f(z) = z^3 - z^2 - 2z + 5$, we have

$$\frac{f(z)}{z-1} = z^2 - 1 + \frac{3}{z-1} \qquad = (z-1)^2 + 2(z-1) - 1 + \frac{3}{z-1}$$

$$\frac{f(z)}{(z-1)^2} = z + 1 + \frac{-1}{z-1} + \frac{3}{(z-1)^2} = (z-1) + 2 + \frac{-1}{z-1} + \frac{3}{(z-1)^2}\,.$$

From the above, for a general analytic function $f(z)$ in a neighborhood of z_0, if we have a Taylor Series (which is NOT known yet, just use as example)

$$f(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots + c_n(z-z_0)^n + \cdots\cdots\,,$$

Then, under reasonable convergence conditions,

$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{c_0}{(z-z_0)^{n+1}} + \cdots + \frac{c_n}{z-z_0} + c_{n+1} + \text{positive powers of } (z-z_0).$$

So we may replace Γ by a small circle with center z_0 to have,

$$\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi\mathbf{i}\,c_n\,.$$

The Cauchy Integral Formula in Theorem 12.1 is the same as the above by observing $c_0 = f(z_0)$. If the above is true, the value of c_n provides the crucial result. Unfortunately, the argument above is not valid because we do not know whether $f^{(n)}(z_0)$ exist for $n \ge 2$ and we are not sure if the Taylor Series converges to the original function. The rigorous proof goes in another direction.

12.2 The Formula

We have seen the intuition above, therefore, we expect the following result.

THEOREM 12.4 (Cauchy Integral Formula). *Let Γ be a simple closed contour with bounded complement component S_b; f be analytic on a domain $\Omega \supset \Gamma \cup S_b$. Then for $0 \le n \in \mathbb{Z}$,*

$$\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \begin{cases} 0 & z_0 \notin S_b\,, \\ 2\pi\mathbf{i}\,\dfrac{f^{(n)}(z_0)}{n!} & z_0 \in S_b\,. \end{cases}$$

Idea of Proof. We will omit the technical steps and focus on the idea. This will let us see the power of the theorem.

We only need to deal with the case that $z_0 \in S_b$. Let us choose a radius $\eta > 0$ such that the ball $B(z_0, \eta) \subset S_b$ and a circle C_δ with center z_0 and radius $\delta < \eta$. For any $z \in B(z_0, \delta)$, by Cauchy Integral Formula in Theorem 12.1 (replacing z_0 by z and integrating wrt ζ), we have

$$f(z) = \frac{1}{2\pi\mathbf{i}} \int_{C_\delta} \frac{f(\zeta)}{\zeta - z}\,d\zeta\,.$$

It can be proved that we can differentiate wrt z on both sides again and again to have,

$$f'(z) = \frac{1}{2\pi\mathbf{i}} \int_{C_\delta} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{f(\zeta)}{\zeta - z}\right] d\zeta = \frac{1}{2\pi\mathbf{i}} \int_{C_\delta} \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta\,,$$

$$f''(z) = \frac{1}{2\pi\mathbf{i}} \int_{C_\delta} \frac{2\,f(\zeta)}{(\zeta - z)^3}\,d\zeta\,,$$

$$f^{(n)}(z) = \frac{1}{2\pi\mathbf{i}} \int_{C_\delta} \frac{n!\,f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta\,. \tag{12.4}$$

In the last line (12.4) above, since the only singularity in the integrand is $z \in S_b$, integration over the contours Γ and C_δ will give the same result,

$$f^{(n)}(z) = \frac{1}{2\pi\mathbf{i}} \int_{C_\delta} \frac{n!\, f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma} \frac{n!\, f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta\,.$$

In particular, this is true for $z = z_0$ and the result is proved. $\square$

Note that in the proof, in order to obtain the result that

$$\int_{C_\delta} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta = 2\pi\mathbf{i}\, \frac{f^{(n)}(z)}{n!}\,,$$

one only needs to assume that f is analytic on $B(z_0, \eta)$ and $|z - z_0| < \delta < \eta$. We can conclude the following surprising fact.

Theorem 12.5. *If f is analytic on $B(z_0, \eta)$ for some $\eta > 0$, then $f', f'', \cdots, f^{(n)}$ exist for all $0 \le n \in \mathbb{Z}$ and are analytic on $B(z_0, \delta)$ for all $\delta < \eta$. Consequently, if f is analytic on a domain Ω, then $f^{(n)}$ exists and is analytic on Ω for all $0 \le n \in \mathbb{Z}$.*

Remark. In the proof, we have omitted the technical step of $\dfrac{\mathrm{d}}{\mathrm{d}z} \displaystyle\int_{C_\delta} g(\zeta, z)\, d\zeta = \int_{C_\delta} \frac{\partial}{\partial z} g(\zeta, z)\, d\zeta$. This is basically due to that $g(\zeta, z)$ is bounded for $\zeta \in C_\delta$ and in the calculation of

$$\lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{C_\delta} \left[\, g(\zeta, z + \Delta z) - g(\zeta, z) \,\right]\, d\zeta\,,$$

the term Δz will be cancelled away.

To this point, we have completed the dotted implications in the diagram of the previous notes.

12.2.1 Average of the Neighbors

Let us re-visit the integral formula and try to view it from the perspective of direct calculation

$$f(z_0) = \frac{1}{2\pi\mathbf{i}} \int_{C_\delta} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \frac{f(z_0 + \delta e^{\mathbf{i}\theta})}{z_0 + \delta e^{\mathbf{i}\theta} - z_0} \left(\delta \mathbf{i} e^{\mathbf{i}\theta}\right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \delta e^{\mathbf{i}\theta})\, d\theta\,.$$

Intuitively, it means that the value of a central point z_0 equals the average value of a small circle (of arbitrary radius) around it. This equation is true on the real u and imaginary v parts of an analytic function. Thus it can also be concluded on harmonic functions.

Theorem 12.6 (Mean Value Property). *If f is analytic or u is harmonic in a ball $B(z_0, \eta)$, then for all $\delta < \eta$,*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \delta e^{\mathbf{i}\theta})\, d\theta \qquad \text{or} \qquad u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \delta e^{\mathbf{i}\theta})\, d\theta\,.$$

The situation of a harmonic function can be figuratively described below.