THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 1 Complex Functions

1.1 Goal and Setting

The theme of our study in this course is ***functions of one complex variable*** and the calculus of such functions. In mathematical notation, the object is

$$f:\ \Omega \subset \mathbb{C} \to \mathbb{C}\,.$$

In this notation, there are three symbols that need further description, namely, f, Ω, and $\mathbb{C}$. We will discuss them one by one.

1.1.1 Complex numbers

Let us start with the set of complex numbers, $\mathbb{C}$, or called the *complex plane*,

$$\mathbb{C} :\stackrel{\text{def}}{=\!=} \{\, z = x + \mathbf{i}y :\ x, y \in \mathbb{R}\,\}\ .$$

Students in this course should know that the addition and multiplication are defined and operated just like real numbers with the inclusion of a special symbol $\mathbf{i}$ and the *working rule* that $\mathbf{i}^2 = -1$. One has all the arithmetic laws of real numbers naturally applicable to complex numbers, nothing surprising.

But, there is an impact. Some polynomial equations with real coefficients may not have real solutions. But, with this new member $\mathbf{i}$, solutions will be created. For example, $\pm\mathbf{i}$ are solutions for the equation $x^2 + 1 = 0$.

1.1.2 Why i is needed?

It is not simply because of wanting to solve the equation $x^2 + 1 = 0$. What is so important to provide a solution for this equation? We all know that there are many situations in mathematics that there is no solution and we are happy with them. ***Why*** are we ***dissatisfied*** with having no solution for this equation?

We have to go back to human history. For many practical purposes, human beings needed to solve for the solution of an equation. The method of solving linear equations was known in many ancient cultures. For the solution to a quadratic equation $ax^2 + bx + c = 0$, we have the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Essentially, this formula was known to the Babylonians in 500 B.C.

Exercise 1.1. To answer the question why we need the symbol $\mathbf{i}$, it would be beneficial for the students to look up Cartan-Tartaglia Formula. Then proceed along the suggestion below.

The Cartan-Tartaglia Formula is an analogous version of

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

It provides a method to find the solutions for a *cubic* (degree 3) or *quartic* (degree 4) polynomial equation. Each cubic or quartic polynomial equation can be reduced to solving a standard form of cubic equation, then the Cartan-Tartaglia Formula can be used. This is known in about 1100 A.D. It is abstractly deduced, so it should give us the roots of the polynomials if they exist.

Related to this issue, some of you may know that Abel shattered the hope of human being and proved that there is no such formula for any polynomial equation of degree ≥ 5. And it latter then leads to the theory of Galois. These are great discoveries in the area of Algebra.

After you have found the Cartan-Tartaglia Formula, you will see that it involves a number of square roots. Use it to solve the equation

$$(x-1)(x-2)(x+3) = 0\,.$$

Obviously, the solutions must be $x = 1, 2, -3$. However, once you have tried the Cartan-Tartaglia Formula on it, you will be surprised that the formula does not give you the answer. Somewhere in the steps, a square root cannot be taken.

In this sense, the Cartan-Tartaglia Formula is relating a real polynomial to its real roots, but it must go through a space that contains square roots of negative numbers.

1.1.3 Would we create more trouble?

The intial difficulty is about finding *real* roots to a polynomial of *real* coefficents. In the process (at least, for degree 3 and 4), we discovered that we need $\mathbf{i}$. For this, we have a large space $\mathbb{C}$. Then, we will have a new type of polynomials, i.e., polynomials with *complex* coefficients. Naturally, we accept *complex* roots to such new type of polynomials. Would we come across a similar situation and need to further expand the space? Would this process continue and have no ending?

The answer is no. It is given by the Fundamental Theorem of Algebra:
Every polynomial of complex coefficient must have a complex root.

The proof of this theorem will be given in the course later, by analytical method using calculus (surprising). In fact, this theorem also has a topological proof. It is indeed a rich theorem not only in the area of Algebra.

With complex numbers, we actually get bonus understanding of geometry and topology. This is a surprising phenomenon. If there is trouble, it is the “curious incident of mathematics” which may cause us sleepless “in the night-time”.

Remark. “Curious Incident of the dog in the Night-time” is an award winning novel, now adapted to a famous stage play, which has been running for two years in London. A live recording will be shown in December in Hong Kong.

1.2 What is Ω?

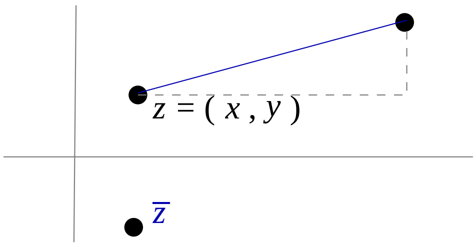
When we talk about a function $f\colon\ \Omega\subset\mathbb{C}\ \to\ \mathbb{C}$, we must talk about the function f and also the domain of interest Ω at the same time. But, before that, we will do something for $\mathbb{C}$ in general.

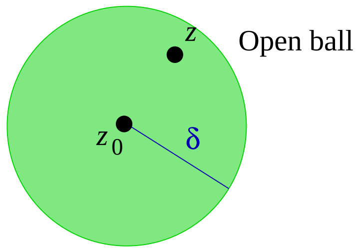
Every complex number $z = x + \mathbf{i}y$ is completely determined by a pair of real numbers, x and y, so it is naturally corresponding to (x, y). Therefore, $\mathbb{C}$ is also represented by the Argand Plane. Note that though it is named after Argand, he is not the first person to use this representation.

Furthermore, for two complex numbers $z_1 = x_1 + \mathbf{i}y_1$ and $z_2 = x_2 + \mathbf{i}y_2$, there distance is

$$|z_1 - z_2| = \sqrt{(z_1 - z_2)\,(\overline{z}_1 - \overline{z}_2)} = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{1/2} .$$

As a result, the geometry of $\mathbb{C}$ is exactly the same as that of the Euclidean plane $\mathbb{R}^2$.

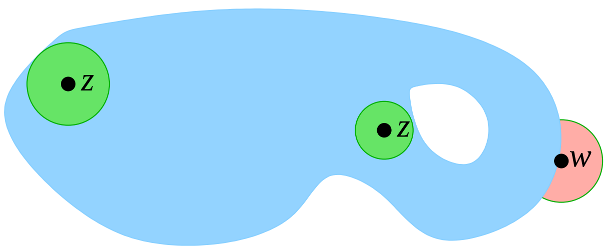




For those who know basic topology, the above distance or metric then defines the topology of $\mathbb{C}$, which again is the same as that of $\mathbb{R}^2$. Here, we will not go into details of topology. Nevertheless, we will introduce the concepts and notations that will be used very often in the future. For example, an *open ball* with center $z_0 \in \mathbb{C}$ and radius $\delta > 0$ is given by

$$B(z_0, \delta) :\overset{\text{def}}{=\!=} \{\, z \in \mathbb{C}:\ \ |z - z_0| < \delta \,\} .$$

This definition of the ball indeed is crucially related to the definition of limit. In the future, we will need to take limit for our function f. Thus, the domain Ω must satisfies one property–if $z_0 \in \Omega$, then a surrounding of z_0 will also belong to Ω. In this way, we can take $f(z)$ for $z \in \Omega$. More precisely, a set Ω is *open* if for each $z_0 \in \Omega$, there is a $\varepsilon > 0$ such that $B(z_0, \varepsilon) \subset \Omega$. Intuitively, the set Ω does not contain its "boundary" and it is to reduce the complication of the limit $z \to z_0$. Observe from the picture below, the two points labelled "z" have a small ball contained in Ω, while the point labelled "w" cannot have such a small ball.

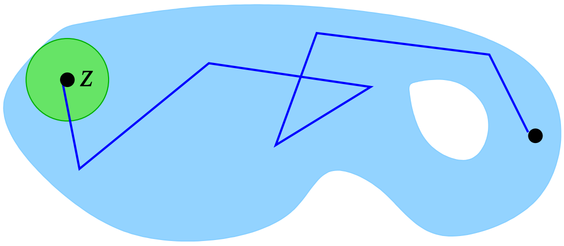


Actually, we only need to use limit to define continuity. Then we basically will not use it anymore. A function $f:\ \Omega \subset \mathbb{C} \to \mathbb{C}$ is *continuous* at $z_0 \in \Omega$ if

$$\lim_{z \to z_0} f(z) = f(z_0) .$$

Or, in the language of ε-δ, if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < |z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$. This, in turns, is equivalent to the following statement about open balls: for every open ball $B(f(z_0), \varepsilon)$, there is an open ball $B(z_0, \delta)$ such that $f(B(z_0, \delta)) \subset B(f(z_0), \varepsilon)$. In this course, we only consider functions f that is continuous at every point in Ω, or simply say, continuous on Ω.

Now, with some basic understanding of the topology of $\mathbb{C}$, it is time to explain the conditions on the subset Ω. There are two of them, *open* and *connected*. We have defined the concept of *open* above. An open set is *connected* if for every pair of points $z_1, z_2 \in \Omega$, there is a finite sequence of straight lines joining from one to another.



This "connectedness" is not a general definition. It is only true for open sets. We require Ω to be connected mostly for convenience and convention.

1.2.1 Typical Examples

While the theory is applicable to all open connected Ω, in this course, most of the Ω actually come from one of the examples below.

1. The whole complex plane, i.e., $\Omega = \mathbb{C}$.
2. The plane with finite points deleted, e.g., $\mathbb{C} \setminus \{0\}$, or a countable discrete set deleted, e.g., $\mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\}$.
3. The plane with a half-line removed, e.g., $\mathbb{C} \setminus \{x : x < 0\}$ or $\mathbb{C} \setminus \{\mathbf{i}y : y > 0\}$.
4. A disk, $B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$ or an annulus,

$$A(z_0, r_1, r_2) = \{ z \in \mathbb{C} : r_1 < |z - z_0| < r_2 \} .$$