Lecture three

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1 Inhomogeneous first order PDE

Example 1. $\frac{du}{dt} = P(t)Q(u)$ then $\int \frac{du}{Q(u)} = \int P(t)dt + C$. For example, $\frac{du}{dt} = -\frac{4}{5}u$.

Proof. From the formula, we have

$$\begin{aligned} \log u &= -\frac{4}{5}t + c \\ u &= e^{c-\frac{4t}{5}}. \end{aligned}$$

□

Example 2. $\frac{du}{dt} + p(t)u = q(t)$.

Proof. Find $f(t)$ such that

$$\begin{aligned} f\frac{du}{dt} + fp(t)u &= fq(t) \\ \frac{d(fu)}{dt} - u\frac{df}{dt} + fp(t)u &= fq(t). \end{aligned}$$

If $\frac{df}{dt} = fp(t)$, we have

$$\begin{aligned} \int \frac{df}{f} &= \int p(t)dt + c \\ \log f &= \int p(t)dt + c. \end{aligned}$$

If we choose $f = e^{\int p(t)dt}$,

$$\frac{d(e^{\int p(t)dt}u)}{dt} = e^{\int p(t)dt}q(t).$$

So we have

$$u(t) = e^{-\int p(t)dt}[\int e^{\int p(t)dt}q(t)dt + c]. \tag{1}$$

For example, if $5u_t + su = st$ we get

$$\begin{aligned} u(t,s) &= e^{-\int \frac{s}{5}dt}[\int e^{\int \frac{s}{5}dt}\frac{st}{5}dt + f(s)] \\ &= e^{-\frac{st}{5}}[\int e^{z}\frac{5z}{s}dz + f(s)] \\ &= e^{-\frac{st}{5}}[te^{\frac{st}{5}} - \frac{5}{s}e^{\frac{st}{5}} + f(s)] \\ &= t - \frac{5}{s} + f(s)e^{-\frac{st}{5}}. \end{aligned}$$

□

Example 3. Solve the inhomogeneous quation

$$u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2.$$

Proof. Method one: Make a change of coordinates

$$t = x + 2y \qquad s = 2x - y.$$

Then we have by the chain rule

$$\begin{aligned} u_x &= u_t + 2u_s \\ u_y &= 2u_t - u_s. \end{aligned}$$

Then

$$\begin{aligned} u_x + 2u_y + (2x-y)u &= u_t + 2u_s + 2(2u_t - u_s) + su \\ &= 5u_t + su \\ &= (2x-y)(x+2y) \\ &= st. \end{aligned}$$

So we get a equation of the form

$$5u_t + su = st.$$

So we have

$$\begin{aligned} u &= t - \frac{5}{s} + f(s)e^{-\frac{st}{5}} \\ &= x + 2y - \frac{5}{2x-y} + f(2x-y)e^{-\frac{2x^2+3xy-2y^2}{5}}, \end{aligned}$$

where f is arbitrary.
Method two: Let

$$z(t) := u(x+t, y+2t).$$

For any fixed x, y, we have the equation

$$\frac{dz}{dt} + (2x-y)z = (2x-y)(x+2y+5t).$$

For fixed x, y, the above equation is a ODE which depends on t. By the formula (1), we can solve z by

$$z(t) - (x+t) - 2(y+2t) + \frac{5}{2x-y} = e^{-(2x-y)t}[z(0) - (x+2y) + \frac{5}{2x-y}].$$

In terms of u, it means

$$u(x+t, y+2t) - [x+t+2(y+2t)] + \frac{5}{2(x+t)-(y+2t)} = e^{-(2x-y)t}[u(x,y) - (x+2y) + \frac{5}{2x-y}].$$

Let $t = -x$, we have

$$u(0, y-2x) - 2(y-2x) - \frac{5}{y-2x} = e^{x(2x-y)}[u(x,y) - (x+2y) + \frac{5}{2x-y}].$$

Denote the left hand side which only depends on $y - 2x$ by

$$\widetilde{f}(y-2x) = u(0, y-2x) - 2(y-2x) - \frac{5}{y-2x}.$$

So the general solution is

$$u(x,y) = x + 2y - \frac{5}{2x-y} + \widetilde{f}(y-2x)e^{-x(2x-y)},$$

for any function $\widetilde{f}$.

We remark that the solution is essentially the same as the solution we get from the first method because the f and $\widetilde{f}$ are arbitary funtions. □

2 Initial and boundary conditions

Because PDEs typically have so many solutions. In real world problems, the PDEs often have additional conditions such as initial conditions and boundary conditions.

An **initial** condition specifies the physical state at a particular time t_0. For example for diffusion equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & -\infty < x < \infty, t > t_0 \\ u(x,t_0) = \phi(x), \end{cases}$$

where $\phi(x)$ is a given function.

For the wave equation there is a pair of initial conditions

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t), \\ u(x,t_0) = \phi(x), \\ u_t(x,t_0) = \psi(x), \end{cases}$$

where $\phi(x)$ is the initial position and $\psi(x)$ is the initial velocity.

In some physical situation, it is necessary to specify some boundary condition if the solution is to be determined. Suppose D is the domain in which the PDE is valid. On the boundary ∂D, the three most important kinds of boundary conditions are:

- u is specified ("Dirichlet condition"),
- the normal derivative $\frac{\partial u}{\partial n}$ is specified ("Neumann condition"), if $\frac{\partial u}{\partial n} = 0$, it is called homogeneous.
- $\frac{\partial u}{\partial n} + au$ is specified ("Robin condition").

For example, the Dirichlet and Neumann boundary of Laplace equation

$$\begin{cases} \triangle u = 0 & \Omega \\ u = \varphi(x). & \partial\Omega \end{cases}$$

and

$$\begin{cases} \triangle u = 1, & \Omega \\ \frac{\partial u}{\partial n} = \varphi(x). & \partial\Omega \end{cases}$$

Example 4. Case 1. Suppose the object D through which the heat is flowing is perfectly insulated, then no heat flows across the boundary and we have the Neumann condition $\frac{\partial u}{\partial n} = 0$.

Case 2. If D was immersed in a large reservoir of specified temperaure $g(t)$ and there were perfect thermal conduction, then we have the Dirichlet condition $u = g(t)$ on ∂D.

Sometimes we need to impose initial condition and boundary condition together as following

$$\begin{cases} u_t = u_{xx}(x,t) & 0 \le x \le l, t > t_0 \\ u(x,t_0) = \phi(x) & 0 \le x \le l \\ u(0,t) = g(t) \\ u(l,t) = h(t), \end{cases} \tag{2}$$

where $\phi(x)$, $g(t)$ and $h(t)$ are known functions.

3 Well-posed problems:

Well-Posed Problems:

1. Existence: There exists at least one solution $u(x,t)$ satisfying all these conditions.
2. Uniqueness: There is at most one solution.
3. Stability: If the data are changed a little, the corresponding solution changed only a little.

The mathematician tries to prove that a given problem is or is not well-posed.

If too few auxiliary conditions are imposed, then there may be more than one solutions (nonuniquess). The problem is called underdetermined.

If too many auxiliary conditions, there may be no solution at all (nonexistence). The problem is called overdetermined.

Stability is usefull in Numerical study. Sometimes, you can't find the solution precisely. You can only measure the data in some approximate sense.

For example:

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & 0 < x < L, \\ u(x.0) = \phi(x), u_t(x,0) = \psi(x), & \\ u(0,t) = g(t), u(L,t) = h(t). & \end{cases}$$

The data of the problem consist of five functions of f, ϕ, ψ, g, h.

Stability means that if f, ϕ, ψ, g, h are perturbed, then u is also changed only slightly.

Example 5. The diffusion equation is well-posed for $t > 0$ and ill-posed for $t < 0$.

Example 6. It is not a well-posed problem to specify both u and $\frac{\partial u}{\partial n}$ on the boundary of D for Laplace equation.

$$u_{xx} + u_{yy} = 0 \quad in \quad D = \{-\infty < x < \infty, 0 < y < \infty\}.$$

One of its solutions is

$$u(x,y) \quad = \tfrac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny.$$

If we prescribed the boundary data to be

$$\begin{aligned} u(x,0) &= 0 \\ \frac{\partial u}{\partial y}(x,0) &= e^{-\sqrt{n}} \sin nx. \end{aligned}$$

If n tends to ∞ then $\frac{\partial u}{\partial y}(x,0)$ tends to zero. But for $y \neq 0$ the solutions $u(x,y)$ do not tend to zero as $n \to \infty$.

4 Types of second order PDE

Let's consider the PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

This is a linear equation of order two in two variables with six real constant coefficients.

Theorem 7. *By a linear transformation of the independent variables, the equation can be reduced to one of three forms, as follows.*

- Elliptic case: if $a_{12}^2 < a_{11}a_{22}$, it is reducible to

  $$u_{xx} + u_{yy} + \cdots = 0$$

  (where $\cdots$ denotes terms of order 1 or 0.

- Hyperbolic case: if $a_{12}^2 > a_{11}a_{22}$, it is reducible to

  $$u_{xx} - u_{yy} + \cdots = 0.$$

- Parabolic case: if $a_{12}^2 = a_{11}a_{22}$,it is reducible to

  $$u_{xx} + \cdots = 0$$

  (unless $a_{11} = a_{12} = a_{22} = 0$).

Definition 8. The PDE

$$\sum_{i,j=1}^{n} a_{ij}u_{ij} + \sum_{i=1}^{n} a_iu_i + a_0u = 0,$$

(with real constants a_{ij}, a_i and a_0. Assume that $a_{ij} = a_{ji}$) is called ELLIPTIC if all the eigenvalues are positive or all are negative. The PDE is called HYPERBOLIC if none of the eigenvalues vanish and one of them has the opposite sign from the $n-1$ others. If none vanish, but at least two of them are positive and at least two are negative, it is called ULTRAHYPERBOLIC. If exactly one of the eigenvalues is zero and all the others have the same sign, the PDE is called PARABOLIC.