Lecture two

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Example 1. Find all solutions satisfy the equation $u_x(x, y) = 0$. If we know the value at $x = x_0$, $u(x_0, y) = y^2$.

Proof. We can integrate once to get $u = constant$ for any fixed y.

$$u(x_1, y) = u(x_2, y).$$

Because u is independent with x. So the solutions are in the form

$$u(x, y) = f(y), \quad (1)$$

where $f(y)$ is arbitrary. If $u(x_0, y) = y^2$, then

$$u(x, y) = y^2.$$

□

Example 2. Solve a constant coefficient transport equation

$$au_x + bu_y = 0, \quad (2)$$

where a and b are constants not both zero.

Proof. Method one (Geometric Method): The quantity $au_x + bu_y$ is the directional derivative of u in the direction of the vector $\overrightarrow{V} = (a, b)$.

This means that $u(x, y)$ must be constant in the direction of $\overrightarrow{V}$. The lines parallel to $\overrightarrow{V}$ have the equations $bx - ay = constant$. They are called the characteristic lines. On any fixed line $bx - ay = c$ the solution u has a constant value. Thus the solution is

$$u(x, y) = f(bx - ay).$$

For example, if $u(0, y) = y^3$ and $a \neq 0$ then

$$u(0, y) = y^3 = f(-ay).$$

Letting $w = -ay$ yields

$$f(w) = -\frac{w^3}{a^3}.$$

So we have

$$u(x,y) = -\frac{(bx-ay)^3}{a^3}.$$

Method two: Make a change of coordinates

$$t = ax + by \qquad s = bx - ay.$$

Replace all x and y derivatives by t and s derivatives. By the Chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = au_t + bu_s,$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = bu_t - au_s.$$

Hence

$$\begin{aligned} au_x + bu_y &= a(au_t + bu_s) + b(bu_t - au_s) \\ &= (a^2+b^2)u_t. \end{aligned}$$

Because $a^2 + b^2 \neq 0$, the equation takes the form $u_t(t,s) = 0$. By the previous example,

$$u(x,y) = u(t,s) = f(s) = f(bx - ay),$$

where f an arbitary function of one variable. □

Example 3. Solve the variable coefficient equation (linear and homogeneous equation)

$$u_x + yu_y = 0.$$

Proof. The same as the geometric method before, the dirctional derivative in the direction of the vector $(1, y)$ is zero. The (characteristic curves) curves in xy plane with $(1, y)$ as its tangent vectors have slopes y. Their equations are

$$\frac{dy}{dx} = \frac{y}{1}.$$

This ODE has the solutions

$$y = se^x.$$

We can check that on each of the curves $u(x, y)$ is a constant becuse

$$\begin{aligned} \frac{d}{dx}u(x, se^x) &= \frac{\partial u}{\partial x} + s\frac{de^x}{dx}\frac{\partial u}{\partial y} \\ &= u_x + se^x u_y \\ &= u_x + yu_y = 0. \end{aligned}$$

The curves fill out the xy plane perfectly without intersecting, as s is changed. Thus suppose (x, y) lies at most and only in one curve (x, se^x)

$$u(x, y) = u(x, se^x) \quad = \quad u(0, s)$$

is independent of x. For any s if we know $u(0, s)$ we know all the value of $u(x, y)$. Putting $s = e^{-x}y$ we have

$$u(x, y) \quad = \quad u(0, e^{-x}y)$$

It follows that

$$u(x, y) \quad = \quad f(e^{-x}y)$$

For example, if $u(0, z) = z^2$, we have

$$u(x, y) \quad = \quad u(0, e^{-x}y) = e^{-2x}y^2.$$

□