Lecture 13

March 3, 2021

In this lecture, we are solving $-X''(x) = \lambda X(x)$ with Robin boundary conditions

$$X' - a_0 X = 0 \quad at \quad x = 0 \tag{1}$$
$$X' + a_l X = 0 \quad at \quad x = l. \tag{2}$$

The two constants a_0 and a_l should be considered as given.

1 Positive eigenvalues

Let's first look for the positive eigenvalues.

$$\lambda = \beta^2 \quad > \quad 0.$$

As usual, the solution of the ODE is

$$X(x) \quad = \quad C\cos\beta x + D\sin\beta x.$$

So that

$$X'(x) \pm aX(x) \quad = \quad (-\beta C \pm aD)\sin\beta x + (\beta D \pm aC)\cos\beta x.$$

From (1) and (2), we have

$$\beta D - a_0 C \quad = \quad 0,$$

and

$$(-\beta C + a_l D)\sin\beta l + (\beta D + a_l C)\cos\beta l \quad = \quad 0.$$

This is linear system with C, D unknowns. Or we can write into the matrix form

$$\begin{bmatrix} -a_0 & \beta \\ -\beta\sin\beta l + a_l\cos\beta l & a_l\sin\beta l + \beta\cos\beta l \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \quad = \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We do not want the trivial solution $C = D = 0$. The coefficient matrix must be zero which is

$$\det \begin{bmatrix} -a_0 & \beta \\ -\beta \sin \beta l + a_l \cos \beta l & a_l \sin \beta l + \beta \cos \beta l \end{bmatrix} = 0.$$

So we have

$$(-a_0 a_l + \beta^2) \sin \beta l = \beta (a_0 + a_l) \cos \beta l.$$

So we get the eigenvalues by finding all the intesection points of the tangent function $y_1(\beta) = \tan \beta l$ and the rational function $y_2(\beta) = \frac{\beta(a_0+a_l)}{\beta^2 - a_0 a_l}$.

Note that the case when $\cos \beta l = 0$ and $\beta^2 = a_0 a_l$ will occur when the graphs of y_1 and y_2 "intersect at infinity" see Figure 1.

One method is to sketch the graphs of $y_1(\beta)$ and the rational function $y_2(\beta)$ as functions of $\beta > 0$ and to find their intersection points.

In positive eigenvalue case, the eigenfunctions are

$$X_n(x) = \cos \beta_n x + \frac{a_0}{\beta_n} \sin \beta_n x.$$

Case 1. $a_0 > 0$ and $a_l > 0$. No matter what they are, as long as they are both positive, Figure 1 or 2 clearly shows that

$$n^2 \frac{\pi^2}{l^2} < \lambda_n < (n+1)^2 \frac{\pi^2}{l^2} \qquad (n = 0, 1, 2, 3, \cdots)$$

and

$$\lim_{n \to \infty} \lambda_n = \frac{n^2 \pi^2}{l^2}.$$

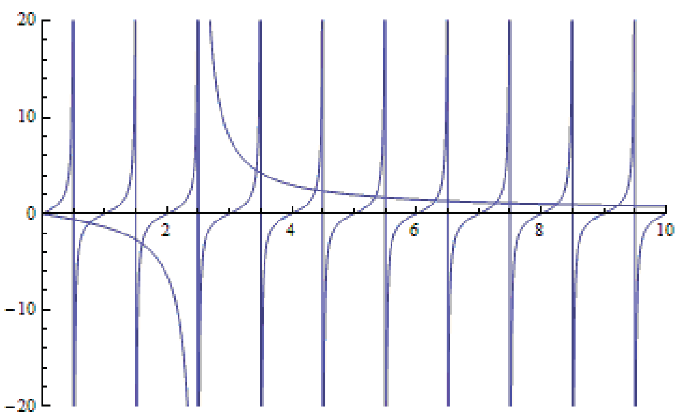

Figure 1: The figure of y_1 and y_2 when $l = \pi, a_0 = 1, a_l = \frac{25}{4}$. This is the case when $\frac{5}{2}$ is one intersection point where the graphs of y_1 and y_2 "intersect at infinity".

The graphs for the more general case looks as following

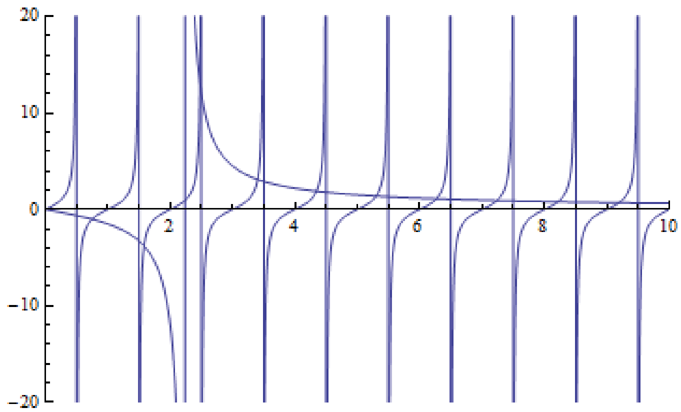

Figure 2: The figure of y_1 and y_2 when $l = \pi, a_0 = 1, a_l = 5$.

Remark 1. For Neumann case, $a_0 = a_l = 0$, the eigenvalues are exactly $\frac{n^2\pi^2}{l^2}$.

Case 2. $a_0 a_l < 0$.

Case 2.1. $a_0 a_l < 0$ and $a_0 + a_l > 0$.

In case $a_0 + a_l > -a_0 a_l l$, the rational curve will start out at the origin with greater slope than the tangent curve and the two graphs must intersect at a point in the interval $(0, \frac{\pi}{2l})$. So there is an eigenvalue $0 < \lambda_0 < (\frac{\pi}{2l})^2$ in this case. The figure in this case is

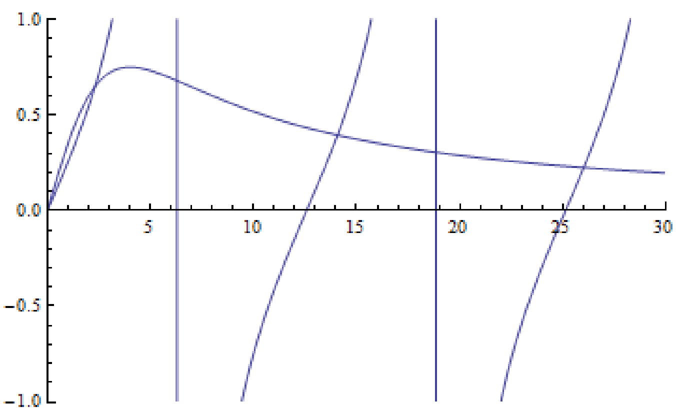

Figure 3: The figure of y_1 and y_2 when $l = \frac{1}{4}, a_0 = 8, a_l = -2$.

In case $a_0 + a_l < -a_0 a_l l$, there is no eigenvalue in $(0, (\frac{\pi}{l})^2)$. The figure in this case is

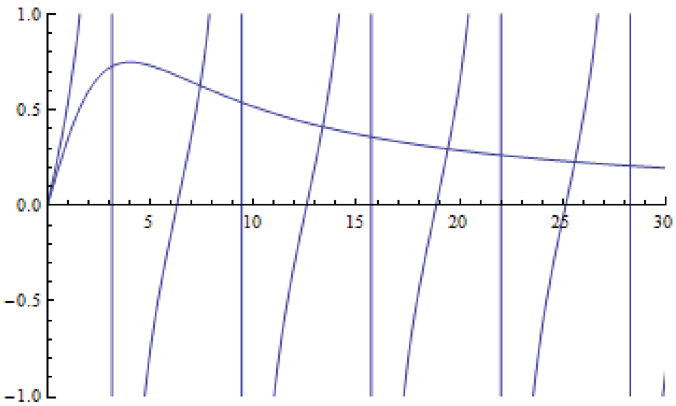

Figure 4: The figure of y_1 and y_2 when $l = \frac{1}{2}, a_0 = 8, a_l = -2$.

Question 2. *What about the case $a_0 a_l < 0$, $a_0 + a_l > 0$ and $a_0 + a_l = -a_0 a_l l$?*

Case 2.2. $a_0 a_l < 0$ and $a_0 + a_l < 0$. The first positive eigenvalue is in $((\frac{\pi}{2l})^2, (\frac{\pi}{l})^2)$. The figure in this case is

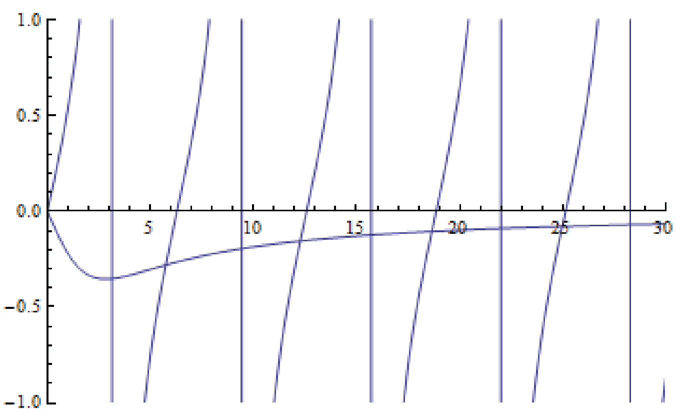

Figure 5: The figure of y_1 and y_2 when $l = \frac{1}{2}, a_0 = 2, a_l = -4$.

Question 3. *What about the case $a_0 a_l < 0$, $a_0 + a_l = 0$?*

Case 3. $a_0 < 0$ and $a_l < 0$.

In case $a_0 + a_l > -a_0 a_l l$, there is no eigenvalue in $(0, (\frac{\pi}{l})^2)$. The figure in this case is

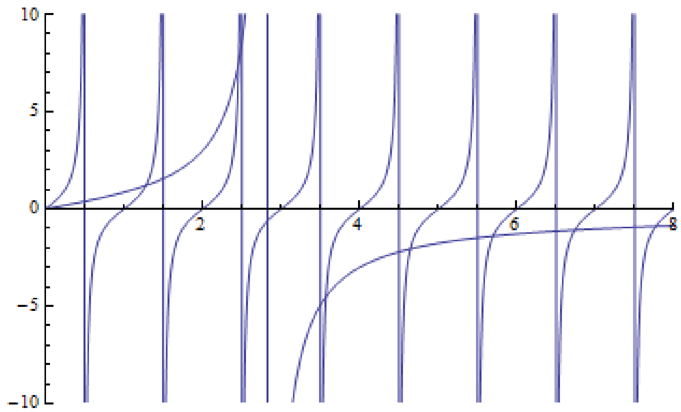

Figure 6: The figure of y_1 and y_2 when $l = \pi, a_0 = -2, a_l = -4$.

In case $a_0 + a_l < -a_0 a_l l$, the first positive eigenvalue is in $(0, (\frac{\pi}{l})^2)$. The figure in this case is

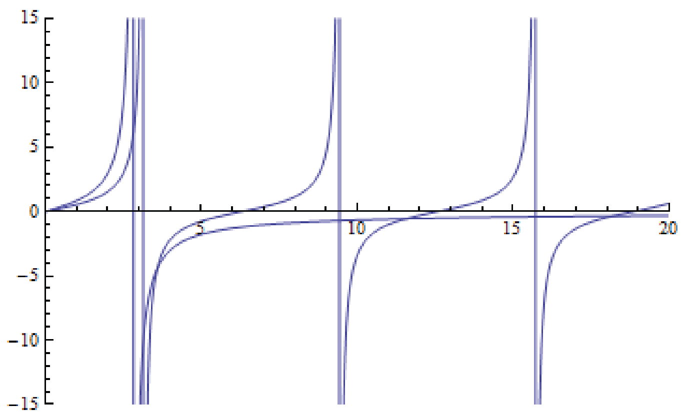

Figure 7: The figure of y_1 and y_2 when $l = \frac{1}{2}, a_0 = -2, a_l = -4$.

Or

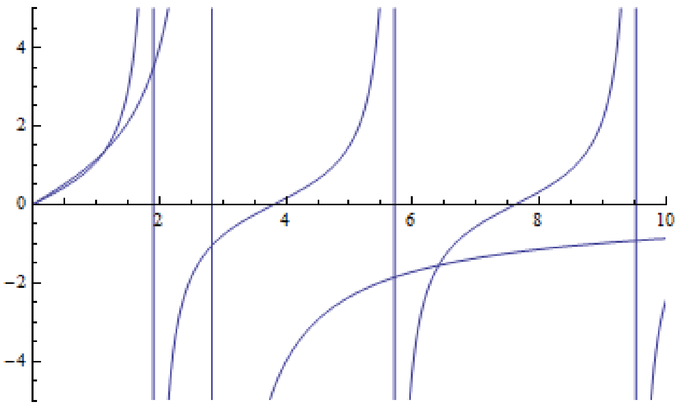

Figure 8: The figure of y_1 and y_2 when $l = \frac{3.3}{4}, a_0 = -4, a_l = -4$.

Question 4. *What about the case $a_0 < 0$, $a_l < 0$ and $a_0 + a_l = -a_0 a_l l$?*

2 Zero eigenvalues

There is a zero eigenvalue if

$$a_0 + a_l = -a_0 a_l l.$$

The eigenfunction is

$$X(x) = 1 + a_0 x.$$

3 Negative eigenvalues

Suppose $-\lambda = \gamma^2 > 0$, the general solution to the equation (1) is

$$X(x) = C \cosh \gamma x + D \sinh \gamma x.$$

Then the boundary conditions (2) and (3) give us a equation for γ

$$\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.$$

In negative eigenvalue case, the eigenfunctions are

$$X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x.$$

So we are looking for intersections of two graphs $z_1(\gamma) = \tanh \gamma l$ and $z_2(\gamma) = -\frac{(a_0+a_l)\gamma}{\gamma^2+a_0 a_l}$.

Case 1. a_0>0 and $a_l > 0$. z_2 will be negative, but z_1 will be positive. So there is no negative eigenvalue in this case.

Case 2. $a_0 a_l < 0$.

Case 2.1. $a_0 a_l < 0$ and $a_0 + a_l > 0$.

In case $a_0 + a_l < -a_0 a_l l$, the slope of z_1 is larger than z_2. So there is exactly one negative eigenvalue. The figure in this case is

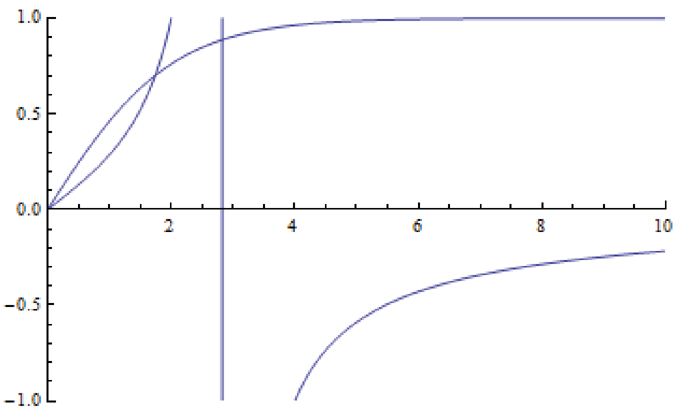

Figure 9: The figure of z_1 and z_2 when $l = \frac{1}{2}, a_0 = 4, a_l = -2$.

In case $a_0 + a_l > -a_0 a_l l$, there is no negative eigenvalue. The figure in this case is

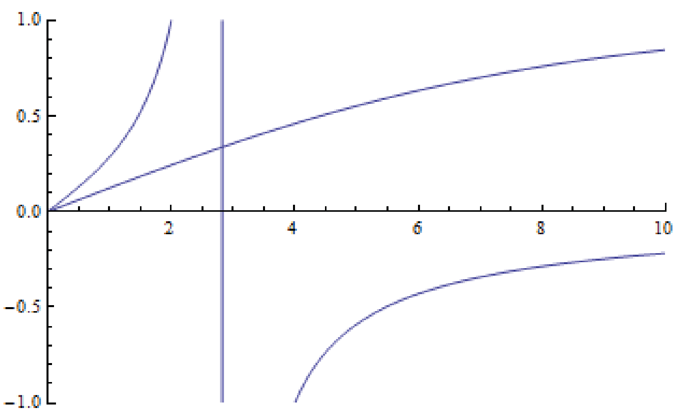

Figure 10: The figure of z_1 and z_2 when $l = \frac{1}{8}, a_0 = 4, a_l = -2$.

Question 5. *What about the case $a_0 a_l < 0$, $a_0 + a_l > 0$ and $a_0 + a_l = -a_0 a_l l$?*

Case 2.2. $a_0 a_l < 0$ and $a_0 + a_l < 0$. There is only one negative eigenvalue. Note in this case, we always have $a_0 + a_l < -a_0 a_l l$. The figure in this case is

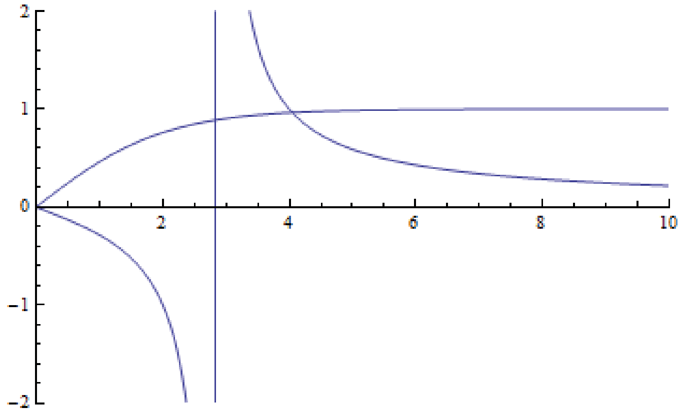

Figure 11: The figure of z_1 and z_2 when $l = \frac{1}{2}, a_0 = 2, a_l = -4$.

Question 6. *What about the case* $a_0 a_l < 0$*,* $a_0 + a_l = 0$*?*

Case 3. $a_0 < 0$ and $a_l < 0$.

In case $a_0 + a_l < -a_0 a_l l$, there is one negative eigenvalue. The figure in this case is

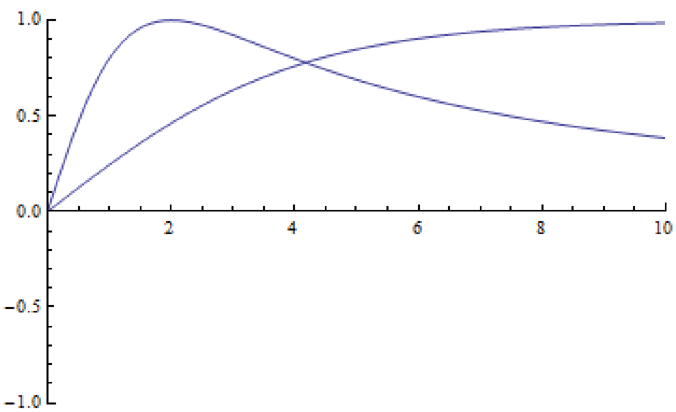

Figure 12: The figure of z_1 and z_2 when $l = \frac{1}{4}, a_0 = -2, a_l = -2$.

In case $a_0 + a_l > -a_0 a_l l$, there are two negative eigenvalues. This is because the maximum point of z_2 is $-\frac{a_0 + a_l}{2\sqrt{a_0 a_l}}$ which is always bigger than 1 while the value of z_1 is asymptotic to 1 from the below. The figure in this case is

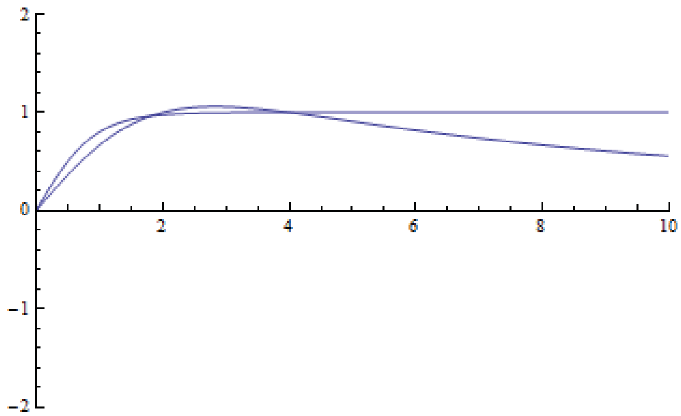

Figure 13: The figure of z_1 and z_2 when $l = 1.1, a_0 = -4, a_l = -2$.

In case $a_0 + a_l = -a_0 a_l l$, there is exactly one negative eigenvalue as show in the Figure

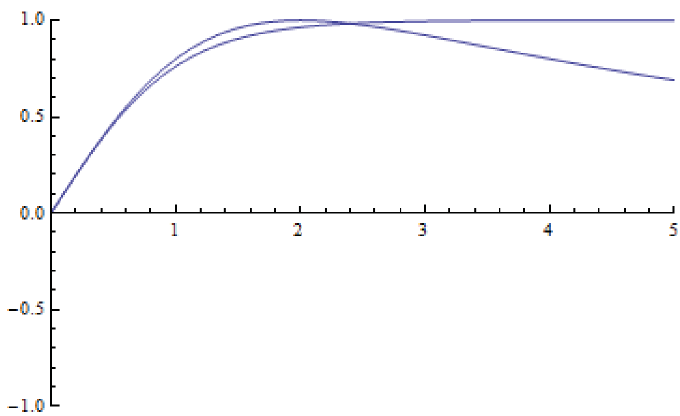

Figure 14: The figure of z_1 and z_2 when $l = 1, a_0 = -2, a_l = -2$.

Summary. Case 1 $a_0 > 0, a_l > 0$: Only positive eigenvalues.

Case 2 $a_0 a_l < 0$ with $a_0 + a_l > -a_0 a_l l$: Only positive eigenvalues.

Case 2 $a_0 a_l < 0$ with $a_0 + a_l < -a_0 a_l l$: One negative eigenvalue, all the rest are positive.

Case 2 $a_0 a_l < 0$ with $a_0 + a_l = -a_0 a_l l$: Zero is an eigenvalue, all the rest are positive.

Case 3 $a_0 < 0, a_l < 0$ with $a_0 + a_l < -a_0 a_l l$: One negative eigenvalue, all the rest are positive.

Case 3 $a_0 < 0, a_l < 0$ with $a_0 + a_l > -a_0 a_l l$: Two negative eigenvalue, all the rest are positive.

Case 3 $a_0 < 0, a_l < 0$ with $a_0 + a_l = -a_0 a_l l$: One negative eigenvalue and one zero eigenvalue, all the rest are positive.

Exercise 7. Analyze the case for $a_0 = 0$ and $a_l \neq 0$.