Lecture one

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1 What is a Partial differential equation (PDE)?

Definition 1. A ***PDE*** is an identity that relates more than one the independent variables (say $x, y, z, t \cdots$), dependent variable $u(x, y, z, \cdots)$, and partial derivatives of u.

- More than one independent variable $x, y, t \cdots$, Ordinary differential equation has only one independent variable x.
- The dependent variable u is an unknown function of these variables $x, y, \cdots$.
- The partial derivatives of u is often denoted by $u_x := \frac{\partial u}{\partial x}$, $u_y := \frac{\partial u}{\partial y}$, $u_{xx} := \frac{\partial^2 u}{\partial x^2}$, $u_{xy} := \frac{\partial^2 u}{\partial x \partial y}$ and so on.

For example: $u_x + u_y = 0$ (transport), $u_{tt} - u_{xx} = 0$ (wave equation), $u_t = u_{xx}$ (diffusion equation) and $u_{xx} + u_{yy} + u_{zz} = 0$ (Laplace's equation).

Let us see the physical interpretation of the above equations.

Example 2. Simple Transport.

Proof. Consider a water flowing at a constant speed c cm/s along a horizontal pipe of fixed cross section in the positive x direction. A pollutant with density $u(x, t)$ g/cm is suspended in the water. The amount of pollutant in the interval $[0, x]$ at time t is $M = \int_0^x u(x', t)dx'$. At the later time $t + h$, the same amount of pollutant have moved to the right by $c \cdot h$ cm. Hence

$$M = \int_0^x u(x', t)dx' = \int_{ch}^{x+ch} u(x', t+h)dx'.$$

Differentiating with respect to x, we get

$$u(x, t) = u(x + ch, t + h).$$

Differentiating with respect to h and putting $h = 0$, we get

$$0 = cu_x(x, t) + u_t(x, t).$$

□

Example 3. Vibrating string.

Proof. A elastic homogeneous string with length l undergoes relatively small transverse vibrations in a plane. Denote $u(x,t)$ to be the hight of the string at time t and position x. Let T (constant) be the magnitude of tension and ρ (constant) be the density (mass per unit length) of string. At very small section of string

$$-T\sin\theta(x,t) + T\sin\theta(x+\triangle x, t) \quad = \quad F.$$

On the other hand, by Newton's law we have

$$F \quad = ma = \quad \rho(\triangle x)u_{tt}.$$

So we have

$$\rho(\triangle x)u_{tt} \quad = \quad -T\sin\theta(x,t) + T\sin\theta(x+\triangle x,t).$$

Dividing both side by $\rho(\triangle x)$,

$$\begin{aligned} u_{tt} &= \frac{T}{\rho}\lim_{\triangle x\to 0}\frac{-\sin\theta(x,t)+\sin\theta(x+\triangle x,t)}{\triangle x} \\ &= \frac{T}{\rho}\frac{\partial}{\partial x}\sin\theta(x,t). \end{aligned}$$

Observing that $u_x = \tan\theta(x,t) \approx \sin\theta(x,t)$, we get

$$u_{tt} \quad = \quad c^2 u_{xx},$$

where $c = \sqrt{\frac{\mathrm{T}}{\rho}}$.

This is the **wave** equation. □

Example 4. Diffusion.

Proof. Let us imagine a motionless liquid filling a straight pipe and a chemical substance which is diffusing through the liquid. Let $u(x,t)$ g/cm be the density of the substance. The mass of it from $[0,x]$ is

$$M \quad = \quad \int_0^x u(x',t)dx'. \tag{1}$$

The chemical substance moves from regions of higher concentration to regions of lower concentration. By Fick's law of diffusion, the rate of motion is proportional to the concentration gradient.

$$\frac{dM}{dt} \quad = \quad flowin - flowout = k(u_x(x,t) - u_x(0,t)), \tag{2}$$

where k is a proportionally constant. So (1) and (2) give the identity

$$\int_0^x u_t(x',t)dx' \quad = \quad k(u_x(x,t) - u_x(0,t)).$$

Differentiating with respect to x, we get

$$u_t = ku_{xx}.$$

This is the **diffusion** equation. □

Example 5. Heat Flow and the Laplace equation.

Proof. Let $u(x, y, z, t)$ be the temperature and $H(t)$ be the amount of heat contained in the region D. Then

$$H(t) = \iiint_D c\rho u dxdy,$$

where c is the "specific heat" of the material and ρ is its density (mass per unit volume). The change of the heat energy in D is

$$\frac{dH}{dt} = \iiint_D c\rho u_t dxdydz \tag{3}$$

On the other hand, Fourier's law says the heat flows from hot to cold regions proportionately to the temperature gradient. But the heat cannot be lost from D except by leaving it through the boundary. This is the law of conservation of energy. Therefore, the change of heat energy in D also equals the heat flux across the boundary,

$$\frac{dH}{dt} = \iint_{\partial D} k(\nabla u \cdot \nu)dS,$$

where k is a heat conductivity and ν is the out normal vector of ∂D.
Denote $\triangle u := u_{xx} + u_{yy} + u_{zz}$.
By divergence theorem,

$$\iint_{\partial D} k(\nabla u \cdot \nu)dS = \iiint_D k\triangle u dxdydz. \tag{4}$$

Thus (3) and (4) give us the **heat** equation

$$u_t = \frac{k}{c\rho}\triangle u.$$

This is the same as the diffusion equation!
In a situation where the physical state does not change with time. Then $u_t = 0$, the heat equation reduce to the Laplace equation

$$\triangle u = 0.$$

For example, the temperature of this room eventually reaches a steady state which satisfies the laplace equation. □

Definition 6. The **order** of an equation is the highest derivative that appears.

For example: $u_x + u_t = 0$ is a first order PDE. $u_{tt} - u_{xx} = 0$ is a second order PDE. $u_{xxx} + u_t + uu_x = 0$ is a third order PDE.

The most general PDE in two independent variables of first order can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0.$$

A ***solution*** *of a PDE is a function* $u(x, y, \cdots)$ *that satisfies the equation identically, at least in some region of the* $x, y, \cdots$ *variables.*

A **operator** $\mathscr{L}$ means: if v is a function $\mathscr{L}v$ is a new function. For instance $\mathscr{L} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is the operator that takes v into $v_x + v_y$.

Definition 7. Linearity: for any functions u and v and any constant c if $\mathscr{L}$ satisfy

$$\mathscr{L}(u + v) = \mathscr{L}u + \mathscr{L}v,$$

and

$$\mathscr{L}(cu) = c\mathscr{L}(u).$$

We call $\mathscr{L}$ is a linear operator.

The equation

$$\mathscr{L}u = 0 \tag{5}$$

is a linear PDE if $\mathscr{L}$ is a linear operator.

Example 8. $u_{xxx} + u_t + uu_x = 0$ is not linear equation. Because the operator $\mathscr{L}u = \frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u$ is not a linear operator.

$$\begin{aligned}
&\mathscr{L}(u + v) = \frac{\partial^3 (u + v)}{\partial x^3} + \frac{\partial}{\partial t}(u + v) + (u + v)\frac{\partial}{\partial x}(u + v), \\
&\mathscr{L}u + \mathscr{L}v = \frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 v}{\partial x^3} + \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}, \\
&\mathscr{L}(u + v) \neq \mathscr{L}u + \mathscr{L}v.
\end{aligned}$$

The equation (5) is called **homogeneous** linear equation. The equation

$$\mathscr{L}u = g,$$

where $g \neq 0$ is a given function of the independent variables, is called an **inhomogeneous** linear equation.

The advantage of linearity for the equation $\mathscr{L}u = 0$ is that

- if u, v are both solutions, so is $au + bv$ for any a and b constants. This is sometimes called the Superposition principle.
- If you add a homogeneous solution to an inhomogeneous solution you get an inhomogeneous solution.