Week 7: Orthogonal complement / Adjoint of operators (textbook §6.2, 6.3)

Orthogonal Complement

Let $(V, \langle , \rangle)$ be an inner product space (could be ∞-dim.)

Def$^{\underline{n}}$: For any subset $\phi \neq S \subseteq V$, we define the orthogonal complement of S to be

$$\boxed{S^{\perp} := \{ x \in V \mid \langle x, y \rangle = 0 \text{ for } \underline{\underline{\text{all}}}\ y \in S \}.}$$

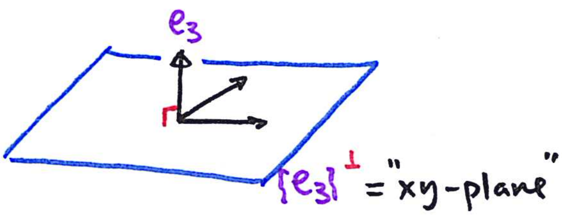
Prop: $S^{\perp}$ is always a linear subspace of V.

(even when S is not a subspace)

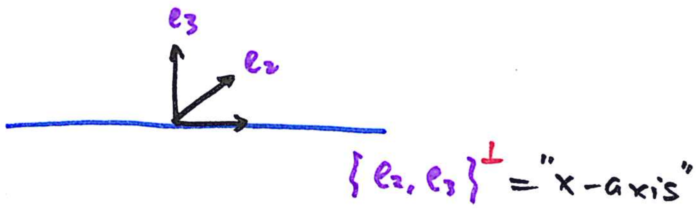
Proof: Exercise!

Examples: $\{0\}^{\perp} = V$, $V^{\perp} = \{0\}$

$\mathbb{R}^3$:



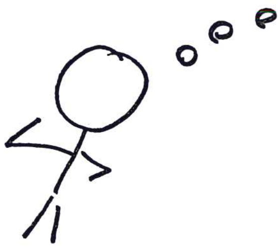
$\mathbb{R}^3$:



Note: $\boxed{S^{\perp} = (\text{Span}\, S)^{\perp}}$ So, WLOG, we can assume S is a subspace!

When S is a subspace of V,

S and $S^{\perp}$ are complementary to each other!



Theorem: Let $W \subseteq V$ be a finite dimensional subspace of an inner product space $(V, \langle , \rangle)$ ← could be ∞-dimensional.

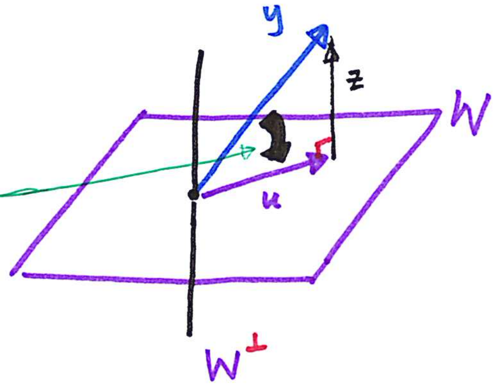
Then, V "splits" into a direct sum:

$$\boxed{V = W \oplus W^{\perp}}$$

In other words, for any $y \in V$, there exist unique $u \in W$ and $z \in W^{\perp}$ s.t.

$$\boxed{y = \overset{\in W}{u} + \overset{\in W^{\perp}}{z}}$$

↑ orthogonal projection of y on W



Proof: "Existence": Pick ANY orthonormal basis $\beta = \{v_1, v_2, \dots, v_k\}$ for W.

For any given $y \in V$, define $\boxed{u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i}$

Clearly, $u \in W$. It remains to check $z := y - u \in W^{\perp}$

$$z \in W^{\perp} \overset{\text{def}}{\iff} \langle z, x \rangle = 0 \quad \text{for all } x \in W$$

$$\underset{\substack{\beta \text{ basis} \\ \text{for } W}}{\iff} \langle z, v_i \rangle = 0 \quad \text{for } i = 1, \dots, k$$

Check:
$$\begin{aligned}
\langle z, v_i \rangle &= \langle y - u, v_i \rangle \\
&= \langle y, v_i \rangle - \langle \overbrace{\sum_{j=1}^{k} \langle y, v_j \rangle v_j}^{u}, v_i \rangle \\
&\quad \swarrow \beta \text{ orthonormal!} \\
&= \langle y, v_i \rangle - \langle y, v_i \rangle = 0.
\end{aligned}$$

Therefore, we have proved

$$y = u + z \in W + W^{\perp}.$$

"Uniqueness": Suffices to show $\boxed{W \cap W^{\perp} = \{0\}}$

[Recall: $V = W_1 \oplus W_2 \iff V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.]

Suppose $x \in W \cap W^{\perp}$. We want to show $x = 0$. Consider

$$\langle \underset{W^{\perp}}{\underset{\cap}{x}}, \underset{W}{\underset{\cap}{x}} \rangle = 0 \quad \text{by def}^{\text{n}} \text{ of } W^{\perp}.$$

$\square$

Geometrically, the orthogonal projection $u \in W$ of $y \in V$ is the unique vector in W which is closest to y.

Pythagoras Theorem: $\boxed{\|x\|^2 + \|y\|^2 = \|x+y\|^2 \text{ when } \langle x, y \rangle = 0}$

Now, we have an orthogonal decomposition:

$$y = u + z \overset{\text{Pythagoras}}{\Longrightarrow} \|u\|^2 + \|z\|^2 = \|y\|^2$$

orthogonal

On the other hand, for any $x \in W$, we have $\underset{W}{\underset{\cap}{u-x}} \perp \underset{W^{\perp}}{\underset{\cap}{z}}$

$$\begin{aligned} \|y - x\|^2 &= \|(u+z) - x\|^2 \\ &= \|(u-x) + z\|^2 \\ &\overset{\text{Pythagoras}}{=} \|u-x\|^2 + \|z\|^2 \\ &\geq \|z\|^2 = \|y-u\|^2. \end{aligned}$$

Therefore, $\boxed{\forall x \in W, \quad \|y - x\| \geq \|y - u\|}$

$\Updownarrow$

"u is the vector in W which is closest to y"
(unique!)

Adjoint of a linear operator

Recall that we have a "dictionary" between linear transformations and matrices (by the choice of bases):

$\mathcal{L}(V,W)$: T, $T \pm U$, cT, $T \circ U$ $\longleftrightarrow$ $M_{m \times n}(\mathbb{F})$: A, $A \pm B$, $c \cdot A$, AB

?? $\longleftrightarrow$ A^t, A^* "transpose"

Ans: adjoint operator T^*

Thm-Def: Let $(V, \langle , \rangle)$ be an inner product space with $\dim V < +\infty$. For any linear operator $T: V \to V$, there exists a unique linear operator $T^*: V \to V$ such that

$$\boxed{\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \text{for } \underline{\underline{\text{all}}} \ x, y \in V}$$

We call T^* the adjoint of T.

Some Formal Properties:

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c}\, T^*$ for any $c \in \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

(c) $(TU)^* = U^* T^*$

(d) $T^{**} = T$

(e) $I^* = I$ where $I: V \to V$ is the identity transformation $I(v) = v$ for $\underline{\underline{\text{all}}}$ $v \in V$.

The properties are like taking (conjugate) transpose of matrices!

Proof: We will prove (c) and (d) below, others are left as exercise.

$\boxed{(c) : (TU)^* = U^* T^*}$ (recall: $\boxed{(AB)^t = B^t A^t}$ for matrices)

For any $x, y \in V$,

$$\langle (TU)x, y \rangle = \langle Ux, T^* y \rangle = \langle x, U^*(T^* y) \rangle .$$

$\boxed{(d) : (T^*)^* = T}$ (recall: $\boxed{(A^t)^t = A}$ for matrices)

For any $x, y \in V$,

$$\langle T^* x, y \rangle = \overline{\langle y, T^* x \rangle} = \overline{\langle Ty, x \rangle} = \langle x, Ty \rangle .$$

$\square$

Question: Why does T^* exist?

To answer this, we need to understand the structure of the space of all linear "functionals" $\mathcal{L}(V, \mathbb{F}) := \{ T : V \to \mathbb{F} \text{ linear} \}$ on a finite dimensional inner product space $(V, \langle , \rangle)$ over $\mathbb{F}$.

For each $v \in V$, we can define a linear functional $T_v : V \to \mathbb{F}$

$$\boxed{T_v(x) := \langle x, v \rangle} \quad \text{for all } x \in V$$

Note: It does not work if we define $x \mapsto \langle v, x \rangle$. Why?

Therefore, we have a map (which is conjugate linear)

$$\begin{array}{ccc} \Phi : V & \xrightarrow{\cong} & \mathcal{L}(V, \mathbb{F}) \\ \psi & & \psi \\ v & \longmapsto & T_v \end{array}$$

It turns out Φ is always one-to-one. (Exercise: prove this!)

When $\dim V < +\infty$, Φ is also onto!

Riesz Representation Theorem:

For a finite dimensional inner product space $(V, \langle , \rangle)$,

the map Φ is a conjugate linear bijective map (i.e. a conjugate linear isomorphism)

In other words, for any linear functional $g : V \to \mathbb{F}$, there exists

a unique vector $y \in V$ s.t. $g = T_y$, i.e.

all linear functionals have this form!

$$\boxed{g(x) = \langle x, y \rangle} \quad \text{for all } x \in V$$

Proof: "Constructive proof" - Find such a vector y explicitly!

① If y were to exist,
then $g(x) = \langle x, y \rangle$ for all $x \in V$

② Expand y in a basis $\boxed{\beta = \{v_1, \dots, v_n\}}$ of V

$$y = a_1 v_1 + \cdots + a_n v_n$$

③ If β is orthonormal, then

$$a_i = \langle y, v_i \rangle$$

④ By ①, $a_i = \overline{\langle v_i, y \rangle} = \overline{g(v_i)}$.

⑤ Hence, we should have

$$\boxed{y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i}$$

flow of ideas

natural

magical

flow of proof

Actual proof goes as follows: given $g : V \to \mathbb{F}$, fix any O.N.B. β

define $\boxed{y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i}$ where $\boxed{\beta = \{v_1, \dots, v_n\}}$ orthonormal basis

check: $g(x) = \langle x, y \rangle$ for all $x \in V$

i.e. $g = T_y : V \to \mathbb{F}$

<u>Recall</u>: Two linear transformations $T, U : V \to W$ are the same as long as they agree on <u>any</u> basis!

Suffices to check: $\boxed{g(v_j) = T_y(v_j)}$ for each $v_j \in \beta$.

$$T_y(v_j) = \langle v_j, y \rangle = \langle v_j, \sum_{i=1}^{n} \overline{g(v_i)} v_i \rangle = g(v_j).$$ <u>DONE!</u>

<u>"Uniqueness"</u>? Suppose $T_y = T_{y'}$. Is $y = y'$?

$$T_y = T_{y'} \iff T_y(x) = T_{y'}(x) \text{ for \underline{all} } x \in V$$
$$\iff \langle x, y \rangle = \langle x, y' \rangle \text{ for \underline{all} } x \in V$$
$$\iff y = y'$$
<u>DONE!</u>

$\square$

Now, we can show why T^* exists and is unique for a finite dimensional inner product space. Given $y \in V$

Consider the "functional" $g : V \to \mathbb{F}$ by

$$g(x) := \langle T(x), y \rangle \quad \text{for any } x \in V$$

<u>Note</u>: (1) g is linear.

(2) g depends on y.

Since $g \in \mathcal{L}(V, \mathbb{F})$, by Riesz Representation Theorem, $g = T_{y'}$ for some <u>unique</u> $y' \in V$, i.e. for <u>any</u> $x \in V$

"box of hope"

$$\langle T(x), y \rangle =: g(x) = T_{y'}(x) := \langle x, y' \rangle \boxed{= \langle x, T^* y \rangle}$$

Define $T^* : V \to V$ by $T^*(y) = y'$

as uniquely defined by the procedure above.

Claim: $T^*: V \to V$ is linear!

Note that T^* is so defined such that it satisfies

$$\star\star \quad \boxed{\langle Tx, y\rangle = \langle x, T^* y\rangle} \quad \text{for all } x, y \in V.$$

Check "linearity":

$$\langle x, T^*(c_1 y_1 + c_2 y_2)\rangle \overset{\star\star}{=} \langle Tx, c_1 y_1 + c_2 y_2\rangle$$

$$= \bar{c}_1 \langle Tx, y_1\rangle + \bar{c}_2 \langle Tx, y_2\rangle$$

$$\overset{\star\star}{=} \bar{c}_1 \langle x, T^* y_1\rangle + \bar{c}_2 \langle x, T^* y_2\rangle$$

$$= \langle x, c_1 T^* y_1 + c_2 T^* y_2\rangle$$

Since the above equality holds for all $x \in V$, we have

$$\boxed{T^*(c_1 y_1 + c_2 y_2) = c_1 T^* y_1 + c_2 T^* y_2} \quad \text{i.e. } T^* \text{ is linear!}$$

□

For the uniqueness of T^* defined by $\star\star$,

suppose there is another $U: V \to V$ linear s.t.

$$\langle Tx, y\rangle = \langle x, Uy\rangle \quad \text{for all } x, y \in V$$

Therefore, together with $\star\star$

$$\langle x, T^* y\rangle = \langle Tx, y\rangle = \langle x, Uy\rangle \quad \text{for all } x, y \in V$$

For each fixed $y \in V$,

fixed

$$\langle x, T^* y\rangle = \langle x, Uy\rangle \quad \text{for all } x \in V$$

$$\Rightarrow \quad T^* y = Uy.$$

But $y \in V$ is "arbitrary", therefore $T^* y = Uy$ for all $y \in V$.

Hence, $T^* = U$.

□

Remark: For an infinite dimensional $(V, \langle , \rangle)$, the adjoint T^* (if exists) of a linear operator $T : V \to V$ is one map s.t

$$(\#) \quad \boxed{\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \text{for } \underline{\underline{\text{all}}} \quad x, y \in V}$$

Fact: If such a T^* exists, then it is linear and unique.

Note: In the essential condition (#) that defines the adjoint T^*, we are moving T from the 1^{st} slot to T^* in the 2^{nd} slot.

In fact it is also equivalent to the following:

$$(\#)' \quad \boxed{\langle x, Ty \rangle = \langle T^* x, y \rangle \quad \text{for } \underline{\underline{\text{all}}} \quad x, y \in V}$$

Exercise: Prove $(\#) \Leftrightarrow (\#)'$.

Question: How is the adjoint T^* related to taking (conjugate) transpose A^* of a matrix?

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_2 \end{pmatrix}$$

What is the adjoint operator T^*?

Fix $\beta = \{e_1, e_2\}$ to be the standard basis, then

$$[T]_\beta = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$ $\underline{\underline{Q}}$: what is $[T^*]_\beta$?

Remember:

$$[T^*]_\beta = \begin{pmatrix} | & | \\ T^*(e_1) & T^*(e_2) \\ | & | \end{pmatrix}$$

Since $\beta = \{e_1, e_2\}$ is orthonormal,

$$\begin{cases} T^*(e_1) = \langle T^*(e_1), e_1 \rangle e_1 + \langle T^*(e_1), e_2 \rangle e_2 \\ T^*(e_2) = \langle T^*(e_2), e_1 \rangle e_1 + \langle T^*(e_2), e_2 \rangle e_2 \end{cases}$$

we can calculate the coefficients explicitly:

$$\langle T^*e_1, e_1 \rangle = \langle e_1, Te_1 \rangle = \begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 1$$

$$\langle T^*e_1, e_2 \rangle = \langle e_1, Te_2 \rangle = \begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}2\\3\end{pmatrix} = 2$$

$$\langle T^*e_2, e_1 \rangle = \langle e_2, Te_1 \rangle = \begin{pmatrix}0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 0$$

$$\langle T^*e_2, e_2 \rangle = \langle e_2, Te_2 \rangle = \begin{pmatrix}0\\1\end{pmatrix} \cdot \begin{pmatrix}2\\3\end{pmatrix} = 3$$

transpose !!

Thus,

$$[T^*]_\beta = \begin{pmatrix} | & | \\ T^*e_1 & T^*e_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = [T]_\beta^t .$$

$\square$

This is ALWAYS true! as long as β is orthonormal basis.

Theorem: Let β be an orthonormal basis for a finite dimensional inner product space $(V, \langle , \rangle)$. Then for any linear operator $T : V \to V$,

$$\boxed{[T^*]_\beta = [T]_\beta^*}$$

Proof: Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be the O.N.B.

$$[T^*]_\beta = A \overset{?}{=} B^* = [T]_\beta^*$$

Look at the (i,j)-th entry of the two matrices:

$$A_{ij} = \langle T^* v_j, v_i \rangle = \langle v_j, T v_i \rangle = \overline{\langle T v_i, v_j \rangle} = \overline{B_{ji}} = B^*_{ij}$$

$B = [T]_\beta$

$\square$

Caution: The relation does not hold if β is not orthonormal.

Back to the previous example, recall that $\beta = \{e_1, e_2\}$ standard basis.

$$[T^*]_\beta = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^t = [T]_\beta^t$$

If β' is another basis which is NOT orthonormal, say

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

then by change of basis, we have

$$\begin{cases} [T^*]_{\beta'} = Q^{-1}[T^*]_\beta Q \\ [T]_{\beta'} = Q^{-1}[T]_\beta Q \end{cases}$$

where $Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ "change of coordinates matrix"

with $Q^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Explicit calculations show:

$$[T^*]_{\beta'} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 2 & 5 \end{pmatrix}$$

NOT transpose!!

$$[T]_{\beta'} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Why does it fail?

We have $[T]_\beta^t = [T^*]_\beta$, β = standard basis

but $[T]_{\beta'}^t = (Q^{-1}[T]_\beta Q)^t = Q^t [T]_\beta^t (Q^{-1})^t$

$$= Q^t [T^*]_\beta (Q^{-1})^t \neq Q^{-1}[T^*]_\beta Q = [T^*]_{\beta'}$$

Notice: If we had $\underbrace{Q^t = Q^{-1}}_{\text{Exercise: when does this happen?}}$, then this becomes "=".