MATH4060 Exercise 6

You are not required to submit your solutions to this Homework.

Revision Exercises.

1. Find the Hadamard factorization of the function $e^{z^2} - 1$. (Hint: First factorize $e^z - 1$.)

2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be entire, with $f(0) = 1$. Let $\mathbb{D}_r$ be the open disc $\{z \in \mathbb{C}\colon |z| < r\}$, and C_r be the boundary of $\mathbb{D}_r$.

   (a) State (without proof) an identity that relates the average of $\log|f|$ on C_r to the zeroes of f. You should indicate the condition(s) on r for which your identity holds.

   (b) Let $n(r)$ be the number of zeroes of f inside the disc $\mathbb{D}_r$. If there exists real numbers $A, B > 0$ such that $|f(z)| \leq Ae^{B|z|^\rho}$ for all $z \in \mathbb{C}$, deduce, using the identity you stated in part (a), an upper estimate for the growth of $n(r)$ as $r \to +\infty$.

3. (a) Let $\{w_k\}_{k=1}^\infty$ be an absolutely summable sequence of complex numbers, with $|w_k| \leq 1/2$ for all $k \in \mathbb{N}$, and let

   $$S := \sum_{k=1}^\infty |w_k| < \infty.$$

   Show that there exists a constant C, depending only on S (and not on the specific sequence $\{w_k\}_{k=1}^\infty$), such that for all $K_1, K_2 \in \mathbb{N}$ with $K_1 < K_2$, we have

   $$\left|\prod_{k=1}^{K_1}(1 - w_k)\right| \leq C \quad \text{and} \quad \left|\prod_{k=1}^{K_2}(1 - w_k) - \prod_{k=1}^{K_1}(1 - w_k)\right| \leq C^2 \sum_{k=K_1+1}^{K_2} |w_k|.$$

   You may use freely without proof the following estimate:

   $$|\mathrm{Log}\,(1 - w)| \leq 2|w| \quad \text{whenever } |w| \leq 1/2.$$

   (Here Log is the principal branch of the logarithm.)

   (b) Let $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$. Let $\Lambda = \{m + n\tau\colon m, n \in \mathbb{Z}\}$ be the lattice generated by 1 and τ, and $\{\tau_k\}_{k=1}^\infty$ be an enumeration of $\Lambda \setminus \{(0,0)\}$. Let

   $$E_2(z) := (1 - z)\exp\left(z + \frac{z^2}{2}\right).$$

   Show that

   $$\sigma(z) := z \prod_{k=1}^\infty E_2\left(\frac{z}{\tau_k}\right)$$

   defines an entire function of z of order 2, has simple zeroes at all points of Λ, and does not vanish anywhere on $\mathbb{C} \setminus \Lambda$. You may use freely without proof the following estimate:

   $$|1 - E_2(z)| \leq 2e|z|^3 \quad \text{whenever } |z| \leq 1/2.$$

4. The Γ function is defined, for $\mathrm{Re}\, s > 0$, by the integral

   $$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

   You may assume known that Γ is holomorphic on the half plane where $\mathrm{Re}\, s > 0$. The goal of this problem is to show that Γ extends meromorphically to $\mathbb{C}$, and that $1/\Gamma$ is entire.

(a) State (without proof) a formula relating $\Gamma(s+1)$ to $\Gamma(s)$, that holds for $\operatorname{Re} s > 0$.

(b) Using the formula you stated in (a), explain how one extends Γ to a meromorphic function on $\mathbb{C}$. You should specify where all the poles of Γ are, and the order of each pole.

(c) State (without proof) a formula relating $\Gamma(s)$ to $\Gamma(1-s)$.

(d) Using the formula you stated in (c), explain why $\dfrac{1}{\Gamma(s)}$ is an entire function of s.

5. Let $\vartheta(t)$ be defined by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

for $t > 0$. By applying the Poisson summation formula, establish a formula relating $\vartheta(t)$ and $\vartheta(1/t)$. You may assume known that the Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi \xi^2}$.

6. Let $\mathbb{D}$ be the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$.

(a) Prove the following version of Schwarz lemma: if $f \colon \mathbb{D} \to \mathbb{D}$ is holomorphic, and $f(0) = 0$, then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. State also what happens when the last inequality is an equality at some $z_0 \in \mathbb{D} \setminus \{0\}$, and justify your claim.

(b) Suppose $h \colon \mathbb{D} \to \mathbb{D}$ is a biholomorphism with $h(0) = 0$. How can you use the result in part (a) to say something interesting about h?

7. Let Ω be an open, connected and simply connected subset of $\mathbb{C}$.

(a) Let f be a holomorphic function on Ω.

(i) Show that f has a primitive on Ω, i.e. there exists a holomorphic function F on Ω such that $F' = f$.

(ii) If, in addition, $f(z) \neq 0$ for every $z \in \Omega$, show that there exists a holomorphic g on Ω such that $e^{g(z)} = f(z)$ for all $z \in \Omega$. Hence show that for any positive integer m, there exists a holomorphic function h on Ω such that $h(z)^m = f(z)$ for all $z \in \Omega$. (g and h are called a logarithm and an m-th root of f on Ω, respectively.)

(b) Suppose $\Omega \neq \mathbb{C}$. Using part (a), or otherwise, construct an injective holomorphic map $h \colon \Omega \to \mathbb{C}$ so that the image $h(\Omega)$ is a bounded subset of $\mathbb{C}$.

8. (a) Let Ω be a connected open subset of $\mathbb{C}$, and $f \colon \Omega \to \mathbb{C}$ be a non-constant function on Ω. Suppose $f_n \colon \Omega \to \mathbb{C}$ is a sequence of injective holomorphic functions on Ω, such that for every compact subset K of Ω, the sequence of functions f_n converges uniformly to a function f on K. Show that f injective on Ω.

(b) Explain, in one sentence, how the result in part (a) is used in the proof of the Riemann mapping theorem.

9. Let $\mathbb{D}$ be the open unit disc in the complex plane centered at the origin. $\mathcal{F}$ be the family of holomorphic functions given by

$$\mathcal{F} = \{f \colon \mathbb{D} \to \mathbb{D} \text{ holomorphic and } f(0) = 0\}.$$

Show that there exists some $g \in \mathcal{F}$ such that

$$g''(0) = \sup_{f \in \mathcal{F}} |f''(0)|,$$

and find the value of this supremum.

Additional Exercises about conformal maps

1. Let $\hat{\mathbb{C}}$ be the extended complex plane $\mathbb{C} \cup \{\infty\}$. A mapping $T: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ is called a Möbius transformation. (Here we interpret $T(\infty) = \infty$ if $c = 0$, and interpret $T(-d/c) = \infty$, $T(\infty) = a/c$ if $c \neq 0$.) Show that the set of all Möbius transformations form a group under composition; in particular, if

$$T(z) = \frac{az+b}{cz+d} \quad \text{and} \quad S(z) = \frac{a'z+b'}{c'z+d'},$$

then

$$(T \circ S)(z) = \frac{Az+B}{Cz+D} \quad \text{where} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

2. A translation is a map of the form $z \mapsto z + b$ for some $b \in \mathbb{C}$. A (complex) dilation is a map of the form $z \mapsto az$ for some $a \in \mathbb{C} \setminus \{0\}$. The map $z \mapsto 1/z$ is called an inversion. Show that any Möbius transformation can be written as compositions of translations, (complex) dilations and inversions.

3. For any quadtuple $(z_1, z_2, z_3, z_4) \in \hat{\mathbb{C}}^4$, we define its cross ratio by

$$[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \in \hat{\mathbb{C}}$$

(This is well-defined if z_1, z_2, z_3, z_4 are all distinct and in $\mathbb{C}$; we then extend by continuity to a continuous function from $\hat{\mathbb{C}}^4$ to $\hat{\mathbb{C}}$. It is called a cross ratio, because it can be written as $\frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}$.)

   (a) Show that for any distinct $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation T such that $T(z_2) = 1$, $T(z_3) = 0$ and $T(z_4) = \infty$. Indeed, such T is given by the cross ratio

$$T(z) = [z, z_2, z_3, z_4].$$

   (b) Show that Möbius transformations preserve the cross ratio; i.e.

$$[S(z_1), S(z_2), S(z_3), S(z_4)] = [z_1, z_2, z_3, z_4]$$

   for any Möbius transformation S and any $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$. (Hint: Either use Question 2, or note that it suffices to show that for fixed distinct $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, we have

$$[z, S(z_2), S(z_3), S(z_4)] = [S^{-1}(z), z_2, z_3, z_4] \quad \text{for all } z \in \hat{\mathbb{C}}.$$

   This last identity follows from part (a), since the right hand side is a Möbius transformation that sends $S(z_2), S(z_3), S(z_4)$ to $1, 0, \infty$ respectively.)

4. A generalized circle in $\hat{\mathbb{C}}$ is either a straight line (including the point $\{\infty\}$), or a circle in $\mathbb{C}$.

   (a) Show that any Möbius transformation maps the real axis into a generalized circle. (Hint: This can be done by a direct computation. Let T be a Möbius transformation. If $w = T(x)$ for some $x \in \mathbb{R}$, then $T^{-1}(w) = \overline{T^{-1}(w)}$. Writing $T^{-1}(w) = \frac{aw+b}{cw+d}$ for some $a, b, c, d \in \mathbb{C}$ shows that w lies on a generalized circle.)

(b) Suppose $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$. Show that the four points lie on a generalized circle, if and only if $[z_1, z_2, z_3, z_4] \in \mathbb{R} \cup \{\infty\}$. (Hint: Without loss of generality assume that the four points z_1, z_2, z_3, z_4 are distinct. Now let

$$T(z) := [z, z_2, z_3, z_4]$$

be the Möbius transformation that maps z_2, z_3, z_4 to $1, 0, \infty$ respectively. Then

$$[z_1, z_2, z_3, z_4] \in \mathbb{R} \cup \{\infty\} \quad \Leftrightarrow \quad T(z_1) \in \mathbb{R} \cup \{\infty\} \quad \Leftrightarrow \quad z_1 \in T^{-1}(\mathbb{R} \cup \{\infty\}),$$

so one just needs to note that $T^{-1}(\mathbb{R} \cup \{\infty\})$ is the generalized circle that passes through z_2, z_3, z_4.)

(c) Show that any Möbius transformation maps generalized circles to generalized circles. (Hint: Use part (b) and that Möbius transformations preserve cross ratios. Alternatively, one can use Question 2. Then one just needs to show, by direct computation, that each of the three basic kinds of Möbius transformations preserves generalized circles. The only difficult case is when $T(z) = 1/z$. But first let C be the circle in $\mathbb{C}$ given by the equation $|z - a|^2 = r^2$ for some $a \in \mathbb{C}$ and $r > 0$. Dividing by $|z|^2$, and expanding, this equation can be rewritten as

$$(|a|^2 - r^2)\left|\frac{1}{z}\right|^2 - \frac{a}{z} - \frac{\bar{a}}{\bar{z}} + 1 = 0.$$

Let $w = 1/z$. Depending on whether $r = |a|$ or not, this is the equation of a straight line or a circle in the w-plane. Similarly, if C is the straight line given by $bz + \bar{b}\bar{z} = c$, then the equation of C can be rewritten as

$$\frac{c}{|z|^2} - \frac{b}{\bar{z}} - \frac{\bar{b}}{z} = 0,$$

which is the equation of a circle or a straight line in the w-plane if $w = 1/z$.)

5. In this question we prove a special case of the so called three-lines lemma, which is useful for the next question.

   Let S be the vertical strip $\{z \in \mathbb{C} \colon a < \operatorname{Re} z < b\}$ for some $a, b \in \mathbb{R}$. Let $f \colon S \to \mathbb{C}$ be a holomorphic function on S, that extends continuously to the closure $\overline{S}$ of S. Suppose f is bounded on S (possibly by some very large constant M). If C is a constant for which $|f(z)| \leq C$ for all z on the boundary of S, show that $|f(z)| \leq C$ for all $z \in S$. (This is a generalization of the maximum modulus principle to an unbounded domain.)

   (Hint: Apply the maximum modulus principle to the function $e^{\varepsilon z^2} f(z)$ on $\overline{S}$ for $\varepsilon > 0$, and then let $\varepsilon \to 0^+$. The whole point here being that $|e^{\varepsilon z^2} f(z)| \to 0$ as $\operatorname{Im} z \to \pm\infty$, whenever $\varepsilon > 0$. You should check that this is indeed the case.)

6. For each $r > 1$, let A_r be the annuli $\{z \in \mathbb{C} \colon 1 < |z| < r\}$. The goal of this question is to show that if r_1, r_2 are both greater than 1, and $r_1 \neq r_2$, then there is no biholomorphic map from A_{r_1} onto A_{r_2}.

   Suppose $f \colon A_{r_1} \to A_{r_2}$ is a biholomorphic map for some $r_1, r_2 > 1$. We will show that $r_1 = r_2$.

   (a) Show that if $\delta > 0$ is sufficiently small, then

   $$\text{either} \quad f(A_{1+\delta}) \subset A_{\sqrt{r_1}}, \quad \text{or} \quad f(A_{1+\delta}) \subset A_{r_1} \setminus \overline{A_{\sqrt{r_1}}}.$$

   In the latter case, by replacing f by r_2/f, we may reduce to the first case. Hence from now on, we assume that $f(A_{1+\delta}) \subset A_{\sqrt{r_1}}$ whenever δ is sufficiently small.

(b) Show (after the renormalization in part (a)) that

$$\lim_{\delta \to 0^+} \left(\max_{|z|=1+\delta} |f(z)| \right) = 1, \quad \text{and} \quad \lim_{\delta \to 0^+} \left(\min_{|z|=r_1-\delta} |f(z)| \right) = r_2.$$

(c) Write Log for the natural logarithm of the positive number. Show that the map $w \mapsto e^w$ maps the vertical strip $S := \{w \in \mathbb{C} \colon 0 < \operatorname{Re} w < \operatorname{Log} r_1\}$ into A_{r_1}.

(d) Part (c) allows us to define a holomorphic map $g \colon S \to A_{r_2}$, by

$$g(w) = f(e^w).$$

Let $\alpha = \dfrac{\operatorname{Log} r_2}{\operatorname{Log} r_1}$. Show that

$$|g(w)| = |e^{\alpha w}| \quad \text{for all } w \in S.$$

(Hint: Apply the three-lines lemma to the bounded holomorphic functions $g(w)/e^{\alpha w}$, and $e^{\alpha w}/g(w)$, on the slightly smaller vertical strip $\{w \in \mathbb{C} \colon \eta < \operatorname{Re} w < \operatorname{Log} r_1 - \eta\}$ than S, and let $\eta \to 0^+$.)

(e) Using part (d), show that there exists a constant c with $|c| = 1$ such that

$$f(e^w) = ce^{\alpha w} \quad \text{for all } w \in S.$$

(f) Show that α is an integer. (Hint: Replace w by $w + 2\pi i$ in the formula in part (e).)

(g) Conclude that

$$f(z) = cz^\alpha \quad \text{for all } z \in A_{r_1}.$$

Since f is injective, it follows that $\alpha = 1$, and hence $r_1 = r_2$.