MATH4060 Exercise 5

Due Date: November 29, 2018.

The questions are from Stein and Shakarchi, *Complex Analysis*, unless otherwise stated.

Chapter 1. Exercise 7.

Chapter 2. Exercise 7. (See Remark at the end of this Homework.)

Chapter 8. Exercise 1, 4, 5, 10, 12, 13.

Additional Exercises.

1. (a) Find the image of the strip $S := \{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < 1\}$ under the conformal map $z \mapsto e^z$. Draw also the images of several horizontal and vertical lines in the strip S under this conformal map.
   (b) Find a biholomorphic map from the strip $\{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < 1\}$ to the upper half plane $\{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$.
   (c) Find a biholomorphic map from the upper half disk $\{z \in \mathbb{C} \colon |z| < 1, \operatorname{Im} z > 0\}$ to the half-strip $\{z \in \mathbb{C} \colon \operatorname{Re} z > 0,\ 0 < \operatorname{Im} z < 1\}$.
2. Find a biholomorphic map from the upper half disk $\{z \in \mathbb{C} \colon |z| < 1, \operatorname{Im} z > 0\}$ onto the unit disk $\{z \in \mathbb{C} \colon |z| < 1\}$. (Hint: First show that the map $f(z) = \frac{1+z}{1-z}$ maps the upper half disk biholomorphically onto the first quadrant $\{x + iy \colon x > 0, y > 0\}$. You may express your answer as the composition of several simple maps.)
3. Find a biholomorphic map from the half-strip $\{z \in \mathbb{C} \colon -\pi/2 < \operatorname{Re} z < \pi/2, \operatorname{Im} z > 0\}$ to the upper half space $\{w \in \mathbb{C} \colon \operatorname{Im} w > 0\}$. (Hint: Use Exercise 5 in Chapter 8.)
4. Let $\mathbb{D}$ be the unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$. Suppose $f \colon \mathbb{D} \to \Omega$ is a biholomorphism from $\mathbb{D}$ onto a domain Ω. Let
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
be the power series expansion of f centered at 0. Show that the area of Ω is given by $\pi \sum_{n=1}^{\infty} n|a_n|^2$. (Hint: First show that the area of Ω is given by $\int_{\mathbb{D}} |f'(z)|^2 dxdy$.)

Remark. In Chapter 2, Exercise 7, the following remark was made: "Moreover, it can be shown that equality $[2|f'(0)| = d]$ holds precisely when f is linear, $f(z) = a_0 + a_1 z$." You are not required to give a proof of this remark in this Homework. But for those who are interested, here is a hint how that could be done.

Method 1. The isodiametric inequality states that if Ω is an (open) set in $\mathbb{R}^2$, then
$$\operatorname{Area}(\Omega) \le \pi \left(\frac{\operatorname{diameter}(\Omega)}{2} \right)^2. \tag{1}$$
In other words, if we fix the diameter of Ω, then its area can only be as large as the open disk of the same diameter. Now apply this to the image $f(\mathbb{D})$ where $f \colon \mathbb{D} \to \mathbb{C}$ satisfies $2|f'(0)| = d$, the diameter of $f(\mathbb{D})$. Then invoke the result in Additional Exercise 3 above to conclude.

We remark that the isodiametric inequality is also true in $\mathbb{R}^n$ for all $n \geq 1$. There is also a characterization of the equality case: the equality in (1) holds precisely when Ω is a disk. One can prove (1) using a technique called Steiner symmetrization (which decreases the diameter of Ω without changing the area of Ω). Alternatively, one can prove (1) using the Brunn-Minkowski inequality, since $\Omega - \Omega$ is contained in a ball of radius d, if d is the diameter of Ω.

Method 2. One can also first prove that the odd part of f is just $f'(0)z$ using Schwarz lemma (applied to $[f(z) - f(-z)]/d$). In other words, $f(z) - f(-z) = 2f'(0)z$. Then show that

$$d = \sup_{z\in\mathbb{D}} |f(z) - f(-z)| = \sup_{z\in\mathbb{D}} \sup_{\theta\in\mathbb{R}} |f(z) - f(e^{i\theta} z)|.$$

More generally, show that if $r \in (0, 1)$, then

$$d = \sup_{|z|=r} \left| \frac{f(z) - f(-z)}{z} \right| = \sup_{|z|=r} \sup_{\theta\in\mathbb{R}} \left| \frac{f(z) - f(e^{i\theta} z)}{z} \right|.$$

So if $w \in \mathbb{D}$, then $g(z) := [f(z) - f(-w)]/2d$ maps w to w, and preserves the disk of radius $|w|$ centered at 0. Then $g'(w)$ has zero imaginary part, so $f'(w)/2d$ has zero imaginary part. This holds for all $w \in \mathbb{D}$, so $f'(w)$ is constant on $\mathbb{D}$, which gives the desired claim.