**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**2018-2019 semester 1 MATH4060**
**week 2 tutorial**

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

# 1 Requisite knowledge

MATH4060 is an advanced course. Basic complex analysis is assumed and some general knowledge of mathematics will be used. Some relevant concepts are listed below, but they are not meant to be exhaustive. Students are expected to have mastered concepts in the list of pre-requisite concepts below, and facts in the list of quasi-requisite concepts will be used with little explanation in the course. Facts in the list of pseudo-requisite concepts are helpful for understanding the course materials.

**pre-requisite** holomorphic function, Cauchy integral formula, calculus of residue

**quasi-requisite** measure theory (Fubini's theorem, Lebesgue's dominated convergence theorem and differentiation under integral sign), point-set topology (e.g. dense, compact), big-O and small-o notations

**pseudo-requisite** <u>manifold theory (e.g. residue's independence of the choice of chart), homology (e.g. integral of closed forms on homologous cycles are equal)</u>

In particular, the following facts from measure theory will be very useful to the course.

**Theorem 1** (Fubini's theorem)**.** Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is measurable. If at least one of

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f| dx \right) dy, \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f| dy \right) dx, \iint_{\mathbb{R}^2} |f| dA$$

is finite, then the following quantities are well defined finite numbers and equal:

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} f dx \right) dy, \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f dx \right) dy, \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f dx \right) dy.$$

*Remark.* The following functions are measurable.

- continuous functions
- countably-piecewise continuous functions, in the sense that the function $f$ is continuous on each $E_n$, where $E_n$ are open / closed sets whose union is $\mathbb{R}^2$

*Remark.* The theorem holds if any integral is replaced by a sum. The general explanation is that sum is an integration against a special measure. A simpler way to understand this is that every sum is an integral of a step function, namely,

$$\sum a_n = \int \sum_n [a_{2n}\chi_{[2n,2n+1]}(x) + a_{2n+1}\chi_{(2n+1,2n+2)}]dx,$$

where the integrand is a countably -piecewise continuous function.

**Theorem 2** (Lebesgue's dominated convergence theorem)**.** Suppose $f_n$ are measureable and they converges to $f$ pointwise. If there exists a $g$ such that $|f_n| \leq g$ and $\int |g|$ is finite, then $\int f_n$ converges to $f$.

One of its multitude of applications is the following.

**Theorem 3** (differentiation under integral sign)**.** Let $K$ be a compact set in $\mathbb{R}^n$ and $I$ be an open interval. Suppose $f : K \times I \to \mathbb{R}$ is continuously differentiable (i.e. it admits a continuously differentiable extension to a neighbourhood of $K \times I$). Then

$$\frac{d}{dt}\int_K f dx = \int_K \frac{\partial}{\partial t} f dx.$$

*Remark.* The assumption can be significantly relaxed to $|\frac{\partial}{\partial t} f(x,t)| \leq g(x)$ for some $g$ with $\int_K g < \infty$. However, the above form suffices for most purposes.

# 2 Review on holomorphic function

Let $U$ be an open set in $\mathbb{C} = \mathbb{R}^2$. Suppose $f : U \to \mathbb{C}$ is continuous. The followings are equivalent.

1. $f$ is complex differentiable everywhere.
2. At every $z_0$ there exists $f'(z_0)$ such that $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)$.
3. Cauchy-Riemann equation holds everywhere.
4. Cauchy integral formula holds everywhere.
5. $f$ is analytic.

We prove (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Leftrightarrow$ (2) and (5) $\Rightarrow$ (1) are trivial.

For (2) $\Leftrightarrow$ (3), use the embedding of $\mathbb{C}$ into $M_{2\times 2}(\mathbb{R})$ defined by $x + iy \mapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ and consider the first column.

For (3) $\Rightarrow$ (4), use Cauchy-Goursat theorem on a keyhole contour and observe that the integral on the keyhole converges to the function value (up to $2\pi i$).

For (4) $\Rightarrow$ (5), shift the reference point to the origin and observe $\frac{1}{z-w} = \frac{1}{w}\sum (z/w)^n$ in Cauchy integral formula ($w$ on contour). Apply Fubini's theorem to pull out $z^n$.