THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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week 13 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness. Bold contents are edits.

1 Arzela-Ascoli theorem

This section mainly follows Sections 7.22-7.25 of [5].

The Arzela-Ascoli theorem is proven here for your reference.

Lemma 1 (diagonal argument)**.** Let $Q = \{q_j\}$ be a countable set and $f_n : Q \to \mathbb{C}$ be a sequence of functions. If $\{f_n(q)\}$ is bounded for each $q \in Q$, then (f_n) has a subsequence (f_{n_k}) that is pointwise convergent.

Proof. Below, both superscripts and subscripts denote indices rather than powers. For a function, say s, defined on a subset of $\mathbb{N}$, both $s(n)$ and s_n denotes the function value of s at n.

A sequence $((f_{n^k_l})_{l\in\mathbb{N}})_{k\in\mathbb{N}}$ of subsequences of $(f_n)_{n\in\mathbb{N}}$, i.e. for each k, $n^k : \mathbb{N} \to \mathbb{N}$ defined by $n^k(l) = n^k_l$ is a strictly increasing function, is constructed inductively such that

1. for each k, $(f_{n^k_l})_{l\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$,
2. for each k, $(f_{n^{k+1}_l})_{l\in\mathbb{N}}$ is a subsequence of $(f_{n^k_l})_{l\in\mathbb{N}}$, i.e. there exists a strictly increasing $l^k : \mathbb{N} \to \mathbb{N}$ such that $n^{k+1} = n^k \circ l^k$, and
3. for each k, $(f_{n^{k+1}_l}(q_j))_{l\in\mathbb{N}}$ converges if $j \le k$.

For $k = 1$, since $(f_n(q_1))$ is bounded, it has a convergent subsequence $(f_{n^1_l}(q_1))$. Suppose, for some k, n^k has been inductively constructed. It suffices to construct n^{k+1} such that $(f_{n^{k+1}_l})_{l\in\mathbb{N}}$ is a subsequence of $(f_{n^k_l})_{l\in\mathbb{N}}$ and $(f_{n^{k+1}_l}(q_{k+1}))_{l\in\mathbb{N}}$ converges. Again by boundedness of $(f_{n^k_l}(q_{k+1}))_{l\in\mathbb{N}}$, it has a convergent subsequence, in other words, there exists some strictly increasing $l^k : \mathbb{N} \to \mathbb{N}$ such that $(f_{n^k_{l(j)}}(q_{k+1}))_{j\in\mathbb{N}}$ converges. Letting $n^{k+1}(j) = n^k(l(j))$ gives the desired subsequence.

Now, a subsequence of $(f_n)_{n\in\mathbb{N}}$ that is eventually a subsequence of each $(f_{n^k_l})_{l\in\mathbb{N}}$ is constructed. Then this subsequence converges pointwise.

Note that $n(k) = n^k(k)$ defines a strictly increasing sequence, because $n^{k+1}_{k+1} > n^k_{k+1} > n^k_k$, and hence $(f_{n^k_k})_{k\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$. It remains to construct, for each k, an increasing $p^k : \mathbb{N} \cap [k, \infty) \to \mathbb{N}$ such that $n_j = n^j_j = n^k \circ p_k$ for $j \ge k$, and hence $(f_{n_j})_{j\ge k}$ is a subsequence of $(f_{n^k_l})_{l\in\mathbb{N}}$. Define, for $j \ge k$, $p^k(j) = (l^k \circ l^{k+2} \ldots \circ l^{j-2} \circ l^{j-1})(j)$. The result then follows. $\square$

Proposition 2. Let K be a compact metric space with a countable dense set Q. Let $f_n : K \to \mathbb{C}$ be an equicontinuous sequence. Suppose $\{f_n(q)\}$ is bounded for each $q \in Q$, then (f_n) has a subsequence that is uniformly convergent.

Proof. By the above lemma, passing to a subsequence if necessary, (f_n) is pointwise convergent on Q. For uniform convergence, it remains to show uniform Cauchy-ness. Fix $\varepsilon > 0$ and suppose $d(x, y) < \delta$, where δ is given by the definition of equicontinuity.

By compactness, finitely many $B_i = B(x_i, \delta/2)$'s cover K, and by density, each B_i contains a $q_i \in Q$, so there exist finitely many $q_i \in Q$'s such that for every $x \in K$, there exists a q_i such that $d(x, q_i) < \delta$.

Since

$$|f_n(x) - f_m(x)| \leq |f_n(q_i) - f_m(q_i)| + |f_n(x) - f_n(q_i)| + |f_m(x) - f_m(q_i)|,$$

where the first term is uniformly bounded by ε if n, m is sufficiently large (choose the largest N corresponding to the q_i's), and the other terms are bounded by ε by equicontinuity. Uniform Cauchy-ness then follows. $\square$

Remark. In fact, every compact metric space has a countable dense set, which can be constructed as follows. For each positive integer n, finitely $B(x_i^n, 1/n)$'s cover K. Then $\{x_i^n\}_{n,i}$ is the desired countable dense set.

2 Covering of $\mathbb{C} \setminus \{0, 1\}$ and Picard's Little Theorem

In this section, a covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$ will be constructed to prove Picard's Little Theorem.

Theorem 3 (Picard's Little Theorem)**.** Let $X = \mathbb{C} \setminus \{0, 1\}$. If $f : \mathbb{C} \to X$ is holomorphic, then it is constant.

We first survey covering space theory as in Chapter 1.3 of [2]. A continuous map $p : \tilde{X} \to X$ is a covering map iff for every for every $x \in X$, $p^{-1}\{x\}$ is discrete and it has a neigbhorhood, called an evenly covered neighborhood, U such that $p^{-1}(U)$ is a disjoint union of open sets, called sheets, each of which is homeomorphic to U via p.

Example 4. Let $X = \mathbb{C} \setminus \{0\}$ and $\tilde{X} = \mathbb{C}$. Then $p : \tilde{X} \to X$ defined by $p(z) = \exp z$ is a covering map.

Let $f : Y \to X$ be continuous. $\tilde{f} : Y \to \tilde{X}$ is said to be a lift iff $f = p\tilde{f}$. Lifts may be viewed as partial inverses of the covering map. For instance, let V be a sheet of preimage of an evenly covered neighborhood U of a point $x_0 \in X$. Then $p^{-1} : U \to V$ is well defined, and is the lift of the inclusion map $i : U \to X$ (defined by $i(x) = x$). Less trivially, the logarithm of a map is a lift.

Example 5. Let Ω be a simply-connected domain. Then every continuous $f : \Omega \to \mathbb{C} \setminus \{0\}$ admits a logarithm $g : \Omega \to \mathbb{C}$ such that $\exp g = f$. g is a lift of f with respect to the covering in Example 4.

Lifts are in general unique and they exist under very mild assumptions. In the two propositions below, let $p : \tilde{X} \to X$ be a covering map and $f : Y \to X$ be continuous; $x_0 \in X$, $\tilde{x}_0 \in p^{-1}\{x_0\}$ and $y_0 \in f^{-1}\{x\}$; and suppose futher that Y is connected.

Proposition 6 (Unique lift property; Theorem 1.34 in [2])**.** Lifts of f that agree at a point are identical.

Proof. Fix two such lifts. Since lifts must map nearby points to the same sheet of an evenly covered neighborhood, points where the lifts agree and do not both form open sets. The result follows from connectedness. □

Proposition 7 (Lifting criterion; Theorem 1.33 in [2])**.** Suppose Y is (connected and) locally path connected. Then f lifts iff the image of every loop in Y based at y_0 is homotopic to the image of a loop in $\tilde{X}$ based at $\tilde{X}_0$.

Sketch of Proof. The homotopy lifting criterion is first established: if f can be lifted, then a homotopy $F : Y \times I \to X$ of f (i.e. $F(y, 0) = f(y)$ for every $y \in Y$) can be lifted. To prove this, it suffices to lift F on $N \times I$ for a neighborhood N of y for each $y \in Y$. Uniqueness implies they can be pasted together to form a global lift. Fix $y \in Y$. Compactness implies there are finitely many neighborhoods N_i's of y and subintervals I_i of I such that $F(N_i \times I_i)$ is mapped to an evenly covered open set. Let N be the intersection of the N_i's. Then F may be lifted on $N \times I_i$ one by one to the corresponding sheets through the local inverses.

The proof of the proposition proceeds as follows. By homotopy lifting criterion, paths in X, which are functions from $I = \{*\} \times I$, can be lifted. Then for each y, there exists a path γ from y_0 to y, and $\tilde{f}(y)$ is defined as the endpoint of the lift of $f(\gamma)$. To show $\tilde{f}$ is well defined, consider an alternative path γ' between y_0 and y. Then γ and γ' form a loop Γ. It suffices to show $f(\Gamma)$ lifts. Indeed, this follows by homotopy lifting criterion, since the assumption implies $f(\Gamma)$ is homotopic to $p(\tilde{h})$ for some loop $\tilde{h}$ in $\tilde{X}$, and $p(\tilde{h})$ lifts to $\tilde{h}$ by definition. □

Corollary 8. If Y is (connected and) locally path connected and simply-connected, then f lifts.

Proof. Every loop in Y is null homotopic, and hence so is its image under f, which is the image of a constant loop in $\tilde{X}$. □

Now, to prove Picard's Little Theorem, it suffices to construct a holomorphic covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$. Then holomorphic maps on $\mathbb{C}$ to X lift to $\mathbb{D}$. Then the theorem follows from Liouville's theorem.

To construct the covering map, some geometric terminology is in place. Hereafter, X denotes $\mathbb{C} \setminus \{0, 1\}$ and let $L = X \cap \{\Im z = 0\}$.

On the unit disc $\mathbb{D}$, points on the unit circle are called ideal points, or points at infinity. Geodesics, substitute of lines, are circular arcs that are normal to the unit circle (again, lines are infinitely large circles), and geodesics are to be distinguished from arcs (non-standard terminology), which will exclusively mean arcs on the unit circles below. Note that inversion with respect to geodesics maps ideal points to ideal points. In terms of notations, for distinct ideal points A, B and C, denote by AB the geodesic through A and B, and by $\overset{\frown}{AB}$ the arc that avoids C (C is omitted since it is often clear from context.).

An ideal triangle is a region bounded by three geodesics that intersect pairwise at the infinity. An ideal polygon is defined analogously.

The covering map is constructed in Section 3 of [4] as follows. (In the tutorial, I said this was in Conway's book. In fact, it should be Lang's.)

Proposition 9. There exists a holomorphic covering map $p : \mathbb{D} \to X$.

Proof. Fix three distinct ideal points A, B, C and let T_0 be the ideal triangle they form. By Riemann mapping theorem, there exists a map p from T_0 to the upper half-plane, with A, B, C mapped to $0, 1, \infty$. Then the boundary of T_0 is mapped to L. By Schwarz reflection principle, p may be extended to the image of T_0 under inversion with respect to its boundary. Let T_1 be the union of T_0 and its image under inversion with respect to its boundary. Then F maps T_1 to X. Note that this may be iterated: suppose p maps T_n to X, and its boundary to L, then p may be extended to T_{n+1}, the union of T_n and its image under inversion with respect to its boundary, and the boundary of T_{n+1} is again mapped to L. Lemma 10 shows the union of all T_n is the unit disc. The result then follows. □

Remark. In the proof above, one needs that the Riemann map extends to the boundary. This is guaranteed for domains with a Jordan boundary. This is known as Caratheodory's Theorem and is Theorem 5.1.1 in [3].

Lemma 10. Every point in the disc is in the image of some iterated reflections of an ideal triangle.

Proof. Observe that fractional linear transformations preserve reflection. By mapping a point to the origin, it suffices to show that every ideal triangle can be iteratively reflected to contain the origin. Observe that an ideal triangle contains the origin iff the longest arc (on the unit circle) through the vertices of the ideal triangle is longer than π. It will be shown that an ideal triangle may be iteratively reflected such that, if the image never contains the origin, the length of the longest arc decreases by at least a constant each time, and the length of the shortest arc is bounded below.

Suppose an ideal triangle T does not contain the origin. It will be reflected about the longest edge (which subtends the longest arc on the unit circle). It suffices to show that the length of the shortest arc of T does not decrease after the reflection, and the length of the longest arc diminishes by the length of the shortest arc of T after the reflection.

Let A, B, C be the vertices of T and $\overset{\frown}{AB}$ be the shortest and $\overset{\frown}{AC}$ be the longest. Let P be the (Euclidean) center of the geodesic (which is a part of an Euclidean circle) through A and C. Then the straight line through A and P is perpendicular to the radius r of the unit circle through A. Let D be the reflection of B about r. It suffices to show the image of B about the geodesic AC, which is the nontrivial intersection of the unit circle with the straight line through A and B, does not lies between $\overset{\frown}{AD}$. This is indeed true because the straight line between B and D is also perpendicular to r, and hence parallel to the one through A and P. Rotating the line about B to make it pass through P tilts the nontrivial intersection with the unit circle away from $\overset{\frown}{AD}$. The result then follows. □

Theorem 3 is then a corollary of the above proposition.

If more algebraic topology is assumed, in particular, the existence of a simply-connected **covering** space **(see the discussion in the section titled *The Classification of Covering Spaces* in [2])**, and a stronger Riemann mapping theorem is assumed, namely that $\mathbb{D}$, $\mathbb{C}$ and $\hat{\mathbb{C}}$ are all the simply-connected Riemann surfaces (Theorem 27.9 in [1]), the existence of p may be argued by ruling out the possilibity that X is covered by $\mathbb{C}$ and $\hat{\mathbb{C}}$. This approach is taken in Sections 27.10-13 in [1].

References

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