THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
Homework 5 solution

1.7 We first show $\left|\frac{w-z}{1-\bar{w}z}\right| = 1$ if either variable lies on the unit circle. Note the conjugation and reciprocation are the same on the unit circle. Suppose $|w| = 1$. Then

$$\left|\frac{w-z}{1-\bar{w}z}\right| = \left|\frac{w-z}{1-z/w}\right| = |w|\left|\frac{w-z}{w-z}\right| = |w| = 1.$$

The other case is similar. Direct computation shows fraction is 0 if $z = w$.

Now, the second part of (a), the whole (bii) and one part of (biii) have been proven. Maximum principle implies (bi) (F is clearly holomorphic on the disc, because the only pole is $1/\bar{w}$, which lies outside the unit disc when w lies inside), and hence the remaining part of (a). Direct computation shows $F \circ F = \text{id}$, and hence (iv) holds (since F is its own inverse) and the remaining of (bii) follows.

2.7 Suppose d is finite, for there is nothing to prove otherwise. Use the equation in the hint. The numerator is bounded above by d and the denominator is bounded below by r^2. The length of contour is bounded above by $2\pi r$, and hence the right-hand side is bounded by $\frac{1}{2\pi}\frac{d}{r}(2\pi r) = \frac{d}{r}$. Therefore, $2|f'(0)| \leq d/r$. The result follows from letting $r \to 1$.

8.1 Recall from Tutorial note 3, $\det Df = |\partial_z f|^2 - |\partial_{\bar{z}} f|^2$, which is nonzero if f is holomorphic and $f' \neq 0$. Then local bijection follows from inverse function theorem.

8.4 Let $f(z) = -i\frac{z+1}{z-1}$. Then f maps the unit disc to the upper half-plane, because f is bijective on the Riemann sphere and its inverse $f^{-1}(z) = \frac{z-i}{z+i}$ maps the upper half-plane to the unit disc, where $|z-i| < |z+i|$ (points in upper half-plane are closer to i than $-i$). Then letting $g(z) = (z-i)^2$, $g \circ f$ is the desired map.

8.5 $f(z) = -\frac{1}{2}(z + 1/z)$. f maps into upper half-plane because $\Im f(re^{i\theta}) = -\frac{1}{2}(r - 1/r)\sin\theta > 0$ if $0 < r < 1$ and $0 < \theta < \pi$. $f'(z) = -\frac{1}{2}(1 - 1/z^2)$, which vanishes only at $z = \pm 1$, which does not lie in the half-disc. It suffices to show f is bijective.

Fix w in the upper half-plane and consider the equation $z^2 + 2wz + 1 = 0$ as hinted. Then if a solution z lies in the half-disc, $f(z) = w$. Let z_1, z_2 be the roots of the equation, Since the product of roots is $z_2 = 1/z_1 = \bar{z}_1/|z_1|^2$, and hence

- one root has modulus ≤ 1, the other ≥ 1; and
- one has imaginary part ≥ 0, the other ≤ 0.

Since the mean of roots is $-w$, which is in the lower half-plane, neither root lies on the unit circle or the real axis, because the otherwise suggests the mean lies on the real axis. Then the mean being the in lower half-plane implies the root with larger modulus is on the lower half-plane because it dominates in the mean, and hence the root with modulus < 1 lies in the half-disc. Since there is exactly one root in the half-disc, the map is bijective.

8.10 Let $G(z) = -i\frac{z+1}{z-1}$. By the same argument in the solution of (8.4), $F \circ G$ maps the disc to the disc and fixes 0. By Schwarz lemma, $(F \circ G)(w) \leq |w|$. Direct computation shows $\frac{z-i}{z+i}$ is the inverse of G. The result then follows.

8.12
a. Suppose f fixes a and b, which are distinct. Let φ_a be the Blaschke factor in (1.7) with $w = a$. Then $\varphi_a \circ f \circ \varphi_a$ fixes 0 and $\varphi_a(b)$. By Schwarz lemma, $\varphi_a \circ f \circ \varphi_a$ is a rotation with a nonzero fixed point, and hence is the identity. The result then follows.

b. Define G as in (8.10) and $H(z) = z + 1$. Then $G^{-1}HG$ has no fixed point.

8.13
a. Schwarz lemma applied to $\psi_{f(w)} \circ f \circ \psi_w$ as hinted ($\psi_w = \psi_w^{-1}$) gives $\rho(f(z), f(w)) = |\psi_{f(w)}(f(z)| \leq |\psi_w(z)| = \rho(z, w)$. The desired inequality then follows by unravelling the definitions. The equation follows by applying the inequality on f^{-1}.

b. Since $\psi_z'(0) = 1 - |z|^2$, the result follows from Schwarz lemma (derivative-at-zero part) applied $\psi_{f(z)} \circ f \circ \psi_z$.

Additional exercise

1. a. The image is the open sector of angle 1 in the first quadrant with the positive real axis as part of the boundary. Horizontal and vertical lines are mapped to rays and arcs centred at 0.

   b. $e^{\pi z}$

   c. $\frac{1}{\pi} \log z$

2. Let $f(z) = \frac{1+z}{1-z}$, $g(z) = z^2$ and $h(z) = \frac{z-i}{z+i}$. It is claimed f maps the upper half-disc to the first quadrant, and hence $h \circ g \circ f$ is the desired map, as g maps the first quadrant to the upper half-plane and h, by Theorem 1.2 of the book, maps the upper half-plane to the unit disc.

   Direct computation shows f maps $-1, 0, 1, i, i/2$ to $0, 1, \infty, i, (3 + 4i)/5$ respectively, and since Mobius transformations preserve circles, f maps the upper half-plane and the unit disc to the upper half-plane and the right-half-plane respectively. Since f is bijective on $\hat{\mathbb{C}}$, the f maps the upper half-disc, which is the intersection of the upper half-plane and the unit disc, to the first quadrant, which is the intersection of the upper half-plane and the right half-plane. Bijectivity follows from that of f on $\hat{\mathbb{C}}$.

3. let $g(z) = \exp(i(z + \pi/2))$ and f be the map defined in 8.5. Then $f \circ g$ is the desired map as g maps the half-strip to the upper half-disc.

4. The hint follows from the last equation in 1.3 of Tutorial note 3 by putting $g = \text{id}$. Then

$$\begin{aligned}
\int_{B(0,R)} |f'(z)|^2 dxdy &= \int_{B(0,R)} f'(z)\overline{f'(z)} dxdy \\
&= \int_0^R \int_0^{2\pi} |f'(re^{it})|^2 r dt dr \\
&= \int_0^R \int_0^{2\pi} \sum_{n,m\geq 0} nm a_n \bar{a}_m r^{n-1} r^{m-1} e^{i(n-m)t} r dt dr \\
&= \sum_{n,m\geq 0} nm a_n \bar{a}_m \int_0^R r^{n+m-1} dr \int_0^{2\pi} e^{i(n-m)t} dt \\
&= \sum_{n,m\geq 0} nm a_n \bar{a}_m \frac{R^{n+m}}{n+m} (2\pi \delta_{n,m}) \\
&= \sum_{n\geq 0} nm |a_n|^2 \frac{R^{2n}}{2n} (2\pi) \\
&= \pi \sum_{n\geq 0} n |a_n|^2 R^{2n}.
\end{aligned}$$

The interchange of order of integral and summation is justified by absolute uniform convergence of the Taylor series of f' on $\overline{B(0,R)}$. Now, the final expression in the above chain of equations is a power series in R^2, which, by Theorem 8.2 of Rudin's *Principles of Mathematical Analysis*, converges to $\pi \sum_{n\geq 0 n|a_n|^2}$. (It is actually not difficult to prove the convergence with bare hands. If $\pi \sum_{n\geq 0} n|a_n|^2 = \infty$, the convergence still holds by monotonicity.) The left-hand side converges to the integral on $\mathbb{D}$, by monotonicity. The result then follows.

remark: Alternatively, appealing to Parseval's theorem applied on $t \mapsto f'(re^{it})$ bypasses the convergence argument, and hence removes the need to integrate on a smaller ball.