SOME PROPERTIES OF TRIGONOMETRIC FUNCTIONS

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We defined earlier the sine and cosine by the following series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(We assumed that the series converges for all $x \in \mathbb{R}$, and can be differentiated term by term.) In particular, we believed that

$$\frac{d}{dx} \sin x = \cos x, \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x.$$

Also, we knew $\sin 0 = 0$, $\cos 0 = 1$, and $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ for all $x \in \mathbb{R}$.

Below we deduce all familiar properties of the sine and cosine from this definition.

The key is the following corollary of the mean value theorem:

Corollary 1. *Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$. If*

$$f'(x) = 0$$

for all $x \in \mathbb{R}$, then f is constant on $\mathbb{R}$; in particular, $f(x) = f(0)$ for all $x \in \mathbb{R}$.

Hence we have:

Proposition 2.

$$\sin^2 x + \cos^2 x = 1 \quad \textit{for all } x \in \mathbb{R}. \tag{1}$$

Proof. Let

$$f(x) = \sin^2 x + \cos^2 x.$$

Then

$$f(0) = \sin^2 0 + \cos^2 0 = 1.$$

Also, f is differentiable on $\mathbb{R}$, and

$$f'(x) = 2 \sin x \cos x + 2 \cos x (-\sin x) = 0$$

for all $x \in \mathbb{R}$. Hence Corollary 1 implies that $f(x) = f(0) = 1$ for all $x \in \mathbb{R}$, which is the desired conclusion. □

Next we will prove the compound angle formula:

Proposition 3. *For all $x, y \in \mathbb{R}$, we have*

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{2}$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \tag{3}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{4}$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y. \tag{5}$$

In fact, it is easy to see that (3) follows from (2) by replacing y by $-y$ (and using that $\sin(-y) = -\sin y$, $\cos(-y) = \cos y$). Similarly, (5) follows from (4) by replacing y by $-y$. Hence it suffices to prove (2) and (4).

To do so, one way is to proceed via the following result about differential equation:

Lemma 4. *Suppose* $f\colon \mathbb{R} \to \mathbb{R}$ *is a twice differentiable function, with*

$$f''(x) + f(x) = 0 \quad \text{for all } x \in \mathbb{R},$$

and

$$f(0) = f'(0) = 0.$$

Then $f(x) = 0$ *for all* $x \in \mathbb{R}$.

It says the zero function is the *only* function that satisfies the conditions of the Lemma.

Proof of Lemma 4. We introduce two auxiliary functions $F\colon \mathbb{R} \to \mathbb{R}$, $G\colon \mathbb{R} \to \mathbb{R}$, such that

$$F(x) = f(x)\cos x - f'(x)\sin x, \quad G(x) = f(x)\sin x + f'(x)\cos x$$

for all $x \in \mathbb{R}$. Then F and G are both differentiable, and

$$F'(x) = -f(x)\sin x + f'(x)(\cos x - \cos x) - f''(x)\sin x = 0$$

$$G'(x) = f(x)\cos x + f'(x)(\sin x - \sin x) + f''(x)\cos x = 0$$

for all $x \in \mathbb{R}$ (the last equalities uses the assumption $f'' + f = 0$). Hence both F and G are constants. Also, by our assumptions on f and f', we have

$$F(0) = f(0)\cos 0 - f'(0)\sin 0 = f(0) = 0$$

and

$$G(0) = f(0)\sin 0 + f'(0)\cos 0 = f'(0) = 0.$$

Hence

$$F(x) = 0 = G(x) \quad \text{for all } x \in \mathbb{R},$$

i.e.

$$\begin{cases} f(x)\cos x - f'(x)\sin x = 0 \\ f(x)\sin x + f'(x)\cos x = 0 \end{cases} \quad \text{for all } x \in \mathbb{R}.$$

We now solve for $f(x)$ by eliminating $f'(x)$: we multiply the first equation by $\cos x$, and the second equation by $\sin x$, and take the sum of the resulting equations. Then

$$(\cos^2 x + \sin^2 x) f(x) = 0$$

for all $x \in \mathbb{R}$, which by (1) implies $f(x) = 0$ for all $x \in \mathbb{R}$, as desired. □

We can now prove (2) and (4).

Proof of (2). Fix $y \in \mathbb{R}$. Let $f(x) = \sin(x+y) - \sin x \cos y - \cos x \sin y$. We want to show that $f(x) = 0$ for all $x \in \mathbb{R}$. To do so, we appeal to our lemma. First,

$$f''(x) = -\sin(x+y) + \sin x \cos y + \cos x \sin y = -f(x)$$

for all $x \in \mathbb{R}$, i.e. $f''(x) + f(x) = 0$ for all $x \in \mathbb{R}$. Next,

$$f(0) = \sin(0+y) - \sin 0 \cos y - \cos 0 \sin y = 0.$$

Also,

$$f'(x) = \cos(x+y) - \cos x \cos y + \sin x \sin y$$

for all $x \in \mathbb{R}$. Hence

$$f'(0) = \cos(0+y) - \cos 0 \cos y + \sin 0 \sin y = 0.$$

By the lemma, it follows that $f(x) = 0$ for all $x \in \mathbb{R}$, as desired. □

The proof of (4) is similar, and left as an exercise.

From Proposition 3 we have the usual double angle formula:

Proposition 5. *For all* $x \in \mathbb{R}$, *we have*

(6) $$\sin(2x) = 2\sin x \cos x$$

(7) $$\cos(2x) = \cos^2 x - \sin^2 x$$

(8) $$\cos(2x) = 2\cos^2 x - 1$$

(9) $$\cos(2x) = 1 - 2\sin^2 x$$

Proof. (6) and (7) just follows from (2) and (4) by setting $y = x$. (8) and (9) follow from (7) and an application of (1). ☐

Hence we have the half-angle formula:

Proposition 6. *For all $x \in \mathbb{R}$, we have*

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{10}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{11}$$

Proof. Just rearrange (8) and (9). ☐

Sometimes the following triple-angle formula are also useful:

Proposition 7. *For all $x \in \mathbb{R}$, we have*

$$\sin(3x) = 3\sin x - 4\sin^3 x \tag{12}$$

$$\cos(3x) = 4\cos^3 x - 3\cos x \tag{13}$$

Proof. For all $x \in \mathbb{R}$, we have (by the compound and double angle formula)

$$\begin{aligned} \sin(3x) &= \sin(x + 2x) \\ &= \sin x \cos 2x + \cos x \sin 2x \\ &= \sin x(1 - 2\sin^2 x) + 2\sin x \cos^2 x \\ &= \sin x(1 - 2\sin^2 x) + 2\sin x(1 - \sin^2 x) \\ &= 3\sin x - 4\sin^3 x, \end{aligned}$$

and

$$\begin{aligned} \cos(3x) &= \cos(x + 2x) \\ &= \cos x \cos 2x - \sin x \sin 2x \\ &= \cos x(2\cos^2 x - 1) - 2\sin^2 x \cos x \\ &= \cos x(2\cos^2 x - 1) - 2(1 - \cos^2 x)\cos x \\ &= 4\cos^3 x - 3\cos x. \end{aligned}$$

☐

Also from Proposition 3, we have the following product-to-sum formula (allowing one to convert the product of two trigonometric functions into a sum):

Proposition 8. *For all $x, y \in \mathbb{R}$, we have*

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y)) \tag{14}$$

$$\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y)) \tag{15}$$

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y)) \tag{16}$$

Proof. (14) follows by averaging (2) and (3). (15) follows by averaging (4) and (5). (16) follows by subtracting (4) from (5), and dividing by 2. ☐

Next, we look at why sine and cosine are periodic. We know $\cos 0 = 1$, and we claim $\cos 2 < 0$: in fact $\cos 2$ is defined by an alternating series, and thus

$$\begin{aligned} \cos 2 &= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \dots \\ &\leq 1 - \frac{2^2}{2!} + \frac{2^4}{4!} \\ &< 0. \end{aligned}$$

Since $\cos x$ is continuous on $\mathbb{R}$, there exists a smallest positive number a such that $\cos a = 0$. (Such a is unique.) We now make the following *definition*:

Definition 1. We define the real number π so that

$$\pi := 2a,$$

where a is as above.

Note that if we adopt this definition, then we will have to *prove* that the circumference of a circle of radius r is $2\pi r$; that we will do when we learn about integration, which allows one to make sense of the length of a curve.

Adopting the above definition of π, we show the following:

Proposition 9.

$$\cos x > 0 \quad \textit{for all } x \in [0, \frac{\pi}{2}), \tag{17}$$

and

$$\sin x > 0 \quad \textit{for all } x \in (0, \frac{\pi}{2}]. \tag{18}$$

Also,

$$\cos \frac{\pi}{2} = 0, \tag{19}$$

and

$$\sin \frac{\pi}{2} = 1 \tag{20}$$

Proof. (19) follows immediately from our definition of π.

Now note that $\frac{\pi}{2}$ is the first positive zero of cos, and that $\cos 0 = 1 > 0$. Hence by continuity of cos, we conclude that (17) holds. In particular, $\frac{d}{dx} \sin x = \cos x > 0$ for $x \in [0, \frac{\pi}{2})$. Hence sin is strictly increasing on $[0, \frac{\pi}{2}]$. Since $\sin 0 = 0$, it follows that (18) holds; in particular, $\sin \frac{\pi}{2} > 0$.

We can now determine the value of $\sin \frac{\pi}{2}$. In fact, from (1), we get

$$\sin^2 \frac{\pi}{2} + \cos^2 \frac{\pi}{2} = 1,$$

so from (19), we get $\sin \frac{\pi}{2} = \pm 1$. But we already knew that $\sin \frac{\pi}{2} > 0$, so $\sin \frac{\pi}{2}$ must be 1 (and not -1). This proves (20). ☐

We can now prove the periodicity of sine and cosine:

Proposition 10. *For all $x \in \mathbb{R}$, we have*

$$\sin(x + \frac{\pi}{2}) = \cos x \tag{21}$$

$$\cos(x + \frac{\pi}{2}) = -\sin x \tag{22}$$

$$\sin(x + \pi) = -\sin x \tag{23}$$

$$\cos(x + \pi) = -\cos x \tag{24}$$

$$\sin(x + 2\pi) = \sin x \tag{25}$$

$$\cos(x + 2\pi) = \cos x \tag{26}$$

Proof. To prove (21), let $f(x) = \sin(x + \frac{\pi}{2}) - \cos x$. We want to invoke Lemma 4. To do so, let's check that

$$f''(x) + f(x) = [-\sin(x + \frac{\pi}{2}) + \cos x] + [\sin(x + \frac{\pi}{2}) - \cos x] = 0$$

for all $x \in \mathbb{R}$. Also,

$$f(0) = \sin \frac{\pi}{2} - \cos 0 = 1 - 1 = 0,$$

$$f'(0) = \cos \frac{\pi}{2} + \sin 0 = 0 + 0 = 0.$$

So one can apply Lemma 4, and conclude that $f(x) = 0$ for all $x \in \mathbb{R}$. This proves (21).

The proof of (22) is similar, and left to the reader.

Finally, (23) follows by first applying (21), and then (22):

$$\sin(x + \pi) = \cos(x + \frac{\pi}{2}) = -\sin x,$$

Similarly one can prove (24). Also, (25) follows from applying (23) twice:

$$\sin(x + 2\pi) = -\sin(x + \pi) = \sin x.$$

Similarly one can prove (26). □

We can now sketch the graph of sin and cos. First, from the periodicity established in the earlier proposition, it suffices to sketch sin and cos on $[0, \frac{\pi}{2}]$. But then this is easy: for instance, to sketch sin, knowing that

$$\sin 0 = 0, \quad \sin\frac{\pi}{2} = 1, \quad \sin x \text{ is continuous for } x \in [0, \frac{\pi}{2}],$$
$$\frac{d}{dx}\sin x = \cos x > 0 \quad \text{for } x \in (0, \frac{\pi}{2}),$$
$$\frac{d^2}{dx^2}\sin x = -\sin x < 0 \quad \text{for } x \in (0, \frac{\pi}{2}),$$

allows us to sketch fairly well the graph of sin on $[0, \frac{\pi}{2}]$; it is strictly increasing and concave there. Similarly one can sketch the graph of cos on $[0, \frac{\pi}{2}]$, hence on the whole $\mathbb{R}$ by periodicity.

Another useful formula is

Proposition 11. *For any $x \in \mathbb{R}$,*

$$\sin(\frac{\pi}{2} - x) = \cos x \tag{27}$$

$$\cos(\frac{\pi}{2} - x) = \sin x \tag{28}$$

Proof. To prove (27), just use (21) with x replaced by $-x$, and use that cos is even:

$$\sin(\frac{\pi}{2} - x) = \cos(-x) = \cos x.$$

Similarly one can prove (28). □

If one wants to calculate the values of sin and cos at some special angles, we can also do so:

Proposition 12.

$$\sin\pi = 0 \tag{29}$$

$$\cos\pi = -1 \tag{30}$$

$$\sin\frac{3\pi}{2} = -1 \tag{31}$$

$$\cos\frac{3\pi}{2} = 0 \tag{32}$$

$$\sin 2\pi = 0 \tag{33}$$

$$\cos 2\pi = 1 \tag{34}$$

Proof. Just use (23) and (24) by setting $x = 0$, $\frac{\pi}{2}$ or π. □

Proposition 13.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \tag{35}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \tag{36}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{37}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \tag{38}$$

$$\sin\frac{\pi}{6} = \frac{1}{2} \tag{39}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \tag{40}$$

Proof. (35) follows from (11), (19) and (18):

$$\sin^2 \frac{\pi}{4} = \frac{1}{2}(1 - \cos \frac{\pi}{2}) = \frac{1}{2}$$

so from (18) we get $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Similarly one can prove (36).

To prove (37), note that from (12) and (29),

$$3 \sin \frac{\pi}{3} - 4 \sin^3 \frac{\pi}{3} = \sin \pi = 0.$$

Hence $\sin \frac{\pi}{3}$ is a root of $3x - 4x^3 = 0$. Now we solve this cubic equation: if $3x - 4x^3 = 0$, then $x = 0$, or $3 - 4x^2 = 0$, so $x = 0, \pm\frac{\sqrt{3}}{2}$. But $\sin \frac{\pi}{3} > 0$ by (18). Hence $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, as desired.

(38) then follows from (1) and (17):

$$\cos^2 \frac{\pi}{6} = 1 - \sin^2 \frac{\pi}{6} = 1 - \frac{3}{4} = \frac{1}{4}$$

so $\cos \frac{\pi}{3} > 0$ implies $\cos \frac{\pi}{3} = \frac{1}{2}$.

Also, (39) follows from (27) and (38):

$$\sin \frac{\pi}{6} = \sin(\frac{\pi}{2} - \frac{\pi}{3}) = \cos \frac{\pi}{3} = \frac{1}{2}.$$

Similarly one can deduce (40). □

In what follows, we deduce some properties of tangent and cotangent: recall

$$\tan x = \frac{\sin x}{\cos x}$$

when $\cos x \neq 0$, and

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

when $\sin x \neq 0$. Also

$$\sec x = \frac{1}{\cos x}$$

when $\cos x \neq 0$, and

$$\csc x = \frac{1}{\sin x}$$

when $\sin x \neq 0$.

Proposition 14.

$$1 + \tan^2 x = \sec^2 x \quad \textit{if } \cos x \neq 0 \tag{41}$$

$$1 + \cot^2 x = \csc^2 x \quad \textit{if } \sin x \neq 0 \tag{42}$$

Proof. To prove (41), note that if $\cos x \neq 0$, then

$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The proof of (42) is similar. □

The following is a compound angle formula for tangent:

Proposition 15. *For all $x, y \in \mathbb{R}$, we have*

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad \textit{if } \cos(x + y) \neq 0,\ \cos x \neq 0 \textit{ and } \cos y \neq 0 \tag{43}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \quad \textit{if } \cos(x - y) \neq 0,\ \cos x \neq 0 \textit{ and } \cos y \neq 0 \tag{44}$$

Proof. To prove (43), note that for $x, y \in \mathbb{R}$, if $\cos(x - y) \neq 0$, then

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

If further $\cos x \neq 0$ and $\cos y \neq 0$, then one divides both the numerator and denominator by $\cos x \cos y$, and obtain (43).

(44) follows from (43) by replacing y by $-y$, and using that $\tan y = -\tan y$. □

The following are formula that expresses $\sin x$, $\cos x$ and $\tan x$ in terms of $\tan \frac{x}{2}$ only. They are useful formula in computing integrals involving trigonometric functions (via a technique called t-substitution).

Proposition 16. *For all $x \in \mathbb{R}$, we have*

$$\tan x = \frac{2\tan \frac{x}{2}}{1-\tan^2 \frac{x}{2}} \quad \textit{if } \cos \frac{x}{2} \neq 0 \textit{ and } \cos x \neq 0 \tag{45}$$

$$\sin x = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \quad \textit{if } \cos \frac{x}{2} \neq 0 \tag{46}$$

$$\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \quad \textit{if } \cos \frac{x}{2} \neq 0 \tag{47}$$

Proof. (45) follows from (43) by replacing both x and y there by $\frac{x}{2}$.

To prove (46), note that if $\cos \frac{x}{2} \neq 0$, we can simplify the denominator of the right hand side, and obtain:

$$\frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{2\tan \frac{x}{2}}{\sec^2 \frac{x}{2}} = 2\tan \frac{x}{2}\cos^2 \frac{x}{2} = 2\sin \frac{x}{2}\cos \frac{x}{2} = \sin x$$

(the last equality following from (6).)

To prove (47), note that if $\cos \frac{x}{2} \neq 0$, we can simplify the denominator of the right hand side, and obtain:

$$\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{1-\tan^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} = (1-\tan^2 \frac{x}{2})\cos^2 \frac{x}{2} = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x$$

(the last equality following from (7).) □

Finally, here are some half-angle formula for tangent:

Proposition 17. *For all $x \in \mathbb{R}$, we have*

$$\tan \frac{x}{2} = \frac{\sin x}{1+\cos x} \quad \textit{if } \cos \frac{x}{2} \neq 0 \tag{48}$$

$$\tan \frac{x}{2} = \frac{1-\cos x}{\sin x} \quad \textit{if } \sin \frac{x}{2} \neq 0 \textit{ and } \cos \frac{x}{2} \neq 0 \tag{49}$$

Proof. To prove (48), note that if $\cos \frac{x}{2} \neq 0$, then by double angle formula (6) and (8), we have

$$\frac{\sin x}{1+\cos x} = \frac{2\sin \frac{x}{2}\cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \tan \frac{x}{2}.$$

To prove (49), note that if $\sin \frac{x}{2} \neq 0$ and $\cos \frac{x}{2} \neq 0$, then by double angle formula (6) and (9), we have

$$\frac{1-\cos x}{\sin x} = \frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2}\cos \frac{x}{2}} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \tan \frac{x}{2}.$$

□

We close by mentioning an alternative approach to all these, via complex numbers. The set of all complex numbers will be denoted by $\mathbb{C}$; it is the set of numbers of the form $a+bi$, where $i^2 = -1$, and $a, b \in \mathbb{R}$. They can be added, subtracted, multiplied and divided. Please refer to any standard text on basic properties of complex numbers.

In this alternative approach, one first shows that one can define a function $\exp\colon \mathbb{C} \to \mathbb{C}$ such that

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for all complex numbers z (in particular, the series converges for all $z \in \mathbb{C}$). Then one verifies that

$$\exp(z)\exp(w) = \exp(z+w)$$

for all complex numbers $z, w \in \mathbb{C}$. (This can be done as in the real case.) Also, one checks that

$$\exp(ix) = \cos x + i\sin x$$

for all real numbers $x \in \mathbb{R}$. (This is the so-called Euler's identity.) It follows that for all $x \in \mathbb{R}$, we have

$$\cos x = \frac{1}{2}(\exp(ix) + \exp(-ix)) \tag{50}$$

and

$$\sin x = \frac{1}{2i}(\exp(ix) - \exp(-ix)). \tag{51}$$

Hence for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
\sin(x+y) &= \frac{1}{2i}(\exp(i(x+y)) - \exp(-i(x+y))) \\
&= \frac{1}{2i}(\exp(ix)\exp(iy) - \exp(-ix)\exp(-iy)) \\
&= \frac{1}{2i}((\cos x + i\sin x)(\cos y + i\sin y) - (\cos x - i\sin x)(\cos y - i\sin y)) \\
&= \frac{1}{2i}(i\cos x\sin y + i\sin x\cos y + i\cos x\sin y + i\sin x\cos y) \\
&= \sin x\cos y + \cos x\sin y,
\end{aligned}$$

as in (2). Similarly one can deduce (3), (4) and (5). One can then deduce the double angle formula, the half-angle formula, etc as before. (In fact, sometimes one turns thing around, and define the sine and cosine of a complex number by formula (51) and (50): in other words, for $z \in \mathbb{C}$, sometimes people define

$$\sin z = \frac{1}{2i}(\exp(iz) - \exp(-iz))$$

and

$$\cos z = \frac{1}{2}(\exp(iz) + \exp(-iz)).$$

Then the compound angle formula continues to hold for this complex sine and cosine, by the same proof we just gave. They also admit the same power series expansions as in the real case:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

But they also have many new properties: the most notable one is that they are no longer bounded by 1 (in fact, one can check that $\cos(iy) = i\cosh y \to \infty$ as $y \to \infty$). You will learn more about these functions in complex analysis.)