AN INTRODUCTION TO POWER SERIES

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We have encountered many series of the form $a_0 + a_1 x + a_2 x^2 + \dots$. For instance, we knew

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{x^k}{k!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{(-1)^k x^{2k}}{(2k)!}$$

We also know that the geometric series

$$1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k = \lim_{N \to \infty} \sum_{k=0}^{N} x^k = \frac{1}{1-x}$$

if $|x| < 1$, and the same series diverges if $|x| > 1$. These are basic examples of what is called a power series, and this is what we study next.

Definition 1. Given a point $c \in \mathbb{R}$, and a sequence of (real or complex) numbers $a_0, a_1, \dots$, one can form a power series centered at c:

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \dots,$$

which is also written as

$$\sum_{k=0}^{\infty} a_k (x-c)^k.$$

For example, the series given before Definition 1 are all power series centered at 0.

We want to think of a power series as a function of x. A power series centered at c will surely converge at $x = c$ (because one is just summing a bunch of zeroes then), but there is no guarantee that the series will converge for any other values x. Nonetheless, by comparing to the geometric series, one can prove:

Theorem 1. *Given any power series*

$$\sum_{k=0}^{\infty} a_k (x-c)^k,$$

there exists a number $R \in [0, \infty]$, such that the series converges for all x with $|x-c| < R$, and diverges for all x with $|x-c| > R$.

Such R is clearly uniquely determined by the power series, and is called the radius of convergence of the power series.

For example, the radius of convergence of the power series

$$1 + x + x^2 + x^3 + \dots$$

is 1, since the series converges for x with $|x-0| < 1$, and diverges for x with $|x-0| > 1$.

We also have:

Theorem 2. *Let R be the radius of convergence of the power series*

$$\sum_{k=0}^{\infty} a_k (x-c)^k.$$

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ *exists, then*

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|};$$

If $\lim_{n\to\infty}|a_n|^{1/n}$ *exists, then*

$$R = \frac{1}{\lim_{n\to\infty}|a_n|^{1/n}}.$$

For instance, consider the radius of convergence of the power series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

There $a_n = \frac{1}{n!}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0,$$

and the radius of the power series is ∞, i.e. the series converges for all $x \in \mathbb{R}$. (This is ultimately why we can define the exponential function using this series!)

Similarly, by computing $\lim_{n\to\infty}|a_n|^{1/n}$, one can prove that the series defining sine and cosine converge, for all $x \in \mathbb{R}$.

Note that the above theorem does not say anything about convergence when $|x-c|$ is exactly equal to R. That is something that must be decided case by case.

Also, it can be shown that if $\lim_{n\to\infty}|a_n|^{1/n}$ exists, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, and the two limits are equal. This guarantees that we will get the same radius of convergence for a power series, regardless of the formula we use.

The good thing about staying within the radius of convergence is not just that the series converge, but also that one can differentiate the series term by term:

Theorem 3. *Suppose R is the radius of convergence of the power series*

$$\sum_{k=0}^{\infty} a_k(x-c)^k,$$

and $B_r(c) = \{x\colon |x-c| < R\}$. Then one can define a function f on $B_r(c)$, by

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k \qquad \text{for all } x \text{ with } |x-c| < R.$$

This function f will be infinitely differentiable on $B_r(c)$; in fact, for all positive integers m, we have

$$\frac{d^m f}{dx^m} = \sum_{k=0}^{\infty} \frac{d^m}{dx^m} a_k(x-c)^k \qquad \text{for all } x \text{ with } |x-c| < R.$$

For instance, this is why one can differentiate e^x, and prove that

$$\frac{d}{dx}e^x = e^x;$$

also, this is why one can differentiate $\sin x$ and $\cos x$, and prove that

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x.$$

Also, recall we knew

$$1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

The above theorem allows us to differentiate both sides, and obtain, successively,

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \frac{1}{(1-x)^2} \quad \text{for } |x| < 1$$

$$1 + 3x + 6x^2 + 10x^3 + 15x^4 + \cdots = \frac{1}{(1-x)^3} \quad \text{for } |x| < 1$$

etc. (These are remarkable identities!)

Another consequence of the above theorem is:

Corollary 4. *Let $f(x)$ be as in Theorem 3. Then the Taylor polynomial of f up to order n at c (which we will denote by $T_{n,c}f(x)$) is just the sum of the first $(n+1)$ terms of the series defining f, i.e.*

$$T_{n,c}f(x) = \sum_{k=0}^{n} a_k (x-c)^k.$$

The proof of the corollary is very easy: one just notes that by the theorem, we have

$$f^{(k)}(c) = k! a_k,$$

so

$$T_{n,c}f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=0}^{n} a_k (x-c)^k.$$

This gives us a very powerful tool in computing Taylor polynomials.

For example, here is how we would compute the Taylor polynomial of $\cos(x^2)$. We knew

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

so

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots$$

It follows that the Taylor polynomial of $\cos(x^2)$ up to order 8 at 0 is

$$1 - \frac{x^4}{2!} + \frac{x^8}{4!}.$$

(It would be quite painful to prove this, by directly computing the derivative of $\cos(x^2)$ up to order 8!)

We can now prove another important identity about the logarithm:

Proposition 5. *For $|x| < 1$, we have*

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$$

In fact, the power series on the right hand side has coefficients

$$a_n = \frac{(-1)^{n-1}}{n}$$

so it radius of convergence is

$$\frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim_{n\to\infty} \frac{n}{n+1}} = 1.$$

This allows us to define a function $f\colon (-1,1) \to \mathbb{R}$, by

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots \qquad \text{for } |x| < 1.$$

We can compute the derivative of $f(x)$ by the above theorem:

$$f'(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \qquad \text{for } |x| < 1.$$

One can sum the right hand side above, since it is a geometric series:

$$f'(x) = \frac{1}{1+x} \qquad \text{for } |x| < 1.$$

Hence

$$f'(x) - \frac{d}{dx}\ln(1+x) = 0 \qquad \text{for } |x| < 1.$$

From the mean-value theorem, it follows that $f(x) - \ln(1+x)$ is a constant on $(-1, 1)$, i.e.

$$f(x) - \ln(1+x) = f(0) - \ln(1+0) = 0, \qquad \text{for } |x| < 1.$$

This proves the desired identity.

One can also multiply, divide, and compose power series, to compute the Taylor polynomial of some more complicated functions. e.g. To compute the Taylor polynomial of $\cos(\sin x)$ up to order 4 at 0, we note:

$$\begin{aligned}
\cos(\sin x) &= 1 - \frac{(\sin x)^2}{2!} + \frac{(\sin x)^4}{4!} - \dots \\
&= 1 - \frac{1}{2!}\left(x - \frac{x^3}{3!} + \dots\right)^2 + \frac{1}{4!}\left(x - \frac{x^3}{3!} + \dots\right)^4 - \dots \\
&= 1 - \frac{1}{2!}\left(x^2 - \frac{2x^4}{3!} + \dots\right) + \frac{1}{4!}\left(x^4 + \dots\right) + \dots \\
&= 1 - \frac{1}{2}x^2 + \left(\frac{1}{3!} + \frac{1}{4!}\right)x^4 + \dots
\end{aligned}$$

so the Taylor polynomial of $\cos(\sin x)$ up to order 4 at 0 is

$$1 - \frac{1}{2}x^2 + \left(\frac{1}{3!} + \frac{1}{4!}\right)x^4.$$

Another example: To compute the Taylor polynomial of $\frac{1}{1-\sin x}$ up to order 3 at 0, we note:

$$\begin{aligned}
\frac{1}{1-\sin x} &= 1 + \sin x + (\sin x)^2 + (\sin x)^3 + \dots \\
&= 1 + \left(x - \frac{x^3}{3!} + \dots\right) + \left(x - \frac{x^3}{3!} + \dots\right)^2 + \left(x - \frac{x^3}{3!} + \dots\right)^3 + \dots \\
&= 1 + \left(x - \frac{x^3}{3!} + \dots\right) + \left(x^2 + \dots\right) + \left(x^3 + \dots\right) \\
&= 1 + x + x^2 + \frac{5}{6}x^3 + \dots
\end{aligned}$$

so the Taylor polynomial of $\frac{1}{1-\sin x}$ up to order 3 at 0 is

$$1 + x + x^2 + \frac{5}{6}x^3.$$

(For those who know about absolute convergence, in the last two examples, we actually made use of the absolute convergence of the power series, to justify the rearrangements of the sums involved. Don't worry about this if you have not seen absolute convergence before.)

The proofs of Theorems 1, 2, 3, and the rigorous justification of the last two examples, are beyond the scope of this course. You will learn more about them in more advanced classes.

The tutor will show you some examples how one computes limits using power series or Taylor's theorem. You will find some practice problems in your homework.